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Triangle-free subgraphs at the triangle-free process

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Abstract

We consider the triangle-free process: given an integer n, start by taking a uniformly random ordering of the edges of the complete n-vertex graph K_n. Then, traverse the ordered edges and add each traversed edge to an (initially empty) evolving graph - unless its addition creates a triangle. We study the evolving graph at around the time where $\Theta(n^{3/2+\varepsilon})$ edges have been traversed for any fixed $\varepsilon \in (0, 10^{-10})$. At that time and for any fixed triangle-free graph F, we give an asymptotically tight estimation of the expected number of copies of F in the evolving graph. For F that is balanced and have density smaller than 2 (e.g., for F that is a cycle of length at least 4), our argument also gives a tight concentration result for the number of copies of F in the evolving graph. Our analysis combines Spencer's original branching process approach for analysing the triangle-free process and the semi-random method.

1 Introduction

In this paper we consider the triangle-free process. This is a random greedy process that generates a triangle-free graph as follows. Given $n \in \mathbb{N}$, take a uniformly random ordering of the edges of the complete n-vertex graph K_n. Here, we take that ordering as follows. Let $\beta : K_n \to [0, 1]$ be chosen uniformly at random; order the edges of K_n according to their *birthtimes* $\beta(f)$ (which are all distinct with probability 1), starting with the edge whose birthtime is smallest. Given the ordering, traverse the ordered edges and add each traversed edge to an evolving (initially empty) triangle-free graph, unless the addition of the edge creates a triangle. When all edges of K_n have been exhausted, the process ends. Denote by $\mathbb{TF}(n)$ the triangle-free graph which is the result of the above process. Further, denote by $\mathbb{TF}(n, p)$ the intersection of $\mathbb{TF}(n)$ with $\{f : \beta(f) \le p\}$.

For a graph F, let X_F be the random variable that counts the number of copies of F in $\mathbb{TF}(n, p)$. We use e_F and v_F to denote respectively the number of edges and vertices in a graph F and set $\mathrm{aut}(F)$ to be the number of automorphisms of F. A graph F is *balanced* if $e_F/v_F \ge e_H/v_H$ for all $H \subseteq F$ with $v_H \ge 1$. We say that an event holds *asymptotically almost surely* (*a.a.s.*) if the probability of the event goes to 1 as $n \to \infty$. For $m_1 = m_1(n)$, $m_2 = m_2(n)$, we write $m_1 \sim m_2$ if m_1/m_2 goes to 1 as $n \to \infty$. Let $\ln n$ denote the natural logarithm of n. Our main result follows.

Theorem 1.1. *Fix a triangle-free graph F and $\varepsilon \in (0, 10^{-10})$. For some $p \sim n^{\varepsilon-1/2}$,*

$$\mathbb{E}[X_F] \sim \frac{v_F!}{aut(F)} \binom{n}{v_F} \left(\frac{\ln n^{\varepsilon}}{n}\right)^{e_F/2}.$$

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Our second result gives a concentration result for X_F, for certain fixed triangle-free graphs F.

Theorem 1.2. *Fix a balanced triangle-free graph F with $e_F/v_F < 2$. Then there exists $0 < \varepsilon_F \leq 10^{-10}$ such that for all $\varepsilon \in (0, \varepsilon_F)$ the following holds. For some $p \sim n^{\varepsilon - 1/2}$, a.a.s.,*

$$X_F \sim \frac{v_F!}{aut(F)} \binom{n}{v_F} \left(\frac{\ln n^{\varepsilon}}{n} \right)^{e_F/2}.$$

One interesting point worth making with respect to Theorem 1.2 is this. Let F be a balanced triangle-free graph with density $e_F/v_F < 2$. Fix $\varepsilon \in (0, \varepsilon_F)$, where ε_F is as guaranteed to exist by Theorem 1.2. Let $p \sim n^{\varepsilon - 1/2}$ be as guaranteed to exist by Theorem 1.2. Consider the random graph $\mathbb{G}(n, m)$, which is chosen uniformly at random from among those n-vertex graphs with exactly $m := \lfloor 2^{-1} n^{3/2} \sqrt{\ln n^{\varepsilon}} \rfloor$ edges. Note that by Theorem 1.2, $\mathbb{TF}(n, p)$ and $\mathbb{G}(n, m)$ a.a.s. has asymptotically the same number of edges. This of course follows directly from our choice of the parameter m. The point is that by standard techniques and by Theorem 1.2, we also have that a.a.s., the number of copies of F in $\mathbb{G}(n, m)$ is asymptotically equal to the number of copies of F in $\mathbb{TF}(n, p)$. Furthermore, $\mathbb{G}(n, m)$ is expected to contain many triangles, and indeed it does contain many triangles a.a.s., whereas $\mathbb{TF}(n, p)$ contains no triangles at all. Therefore, one may argue, at least with respect to the number of copies of fixed balanced triangle-free graphs with density strictly less than 2, that $\mathbb{TF}(n, p)$ "looks like" a uniformly random graph with m edges–only that it has no triangles. A similar point can be made with respect to Theorem 1.1.

1.1 Related results

Erdős, Suen and Winkler [5] were the first to consider the triangle-free process. They proved that the number of edges in $\mathbb{TF}(n)$ is a.a.s. bounded by $\Omega(n^{3/2})$ and $O(n^{3/2} \ln n)$. Spencer [12] showed that for every two reals $a_1, a_2 > 0$, there exists n_0 such that the number of edges in $\mathbb{TF}(n)$ for $n \geq n_0$ is expected to be at least $a_1 n^{3/2}$ and is a.a.s. at most $a_2 n^{3/2} \ln n$. In the same paper, Spencer conjectured that the number of edges in $\mathbb{TF}(n)$ is a.a.s. $\Theta(n^{3/2} \sqrt{\ln n})$. In a recent breakthrough, this conjecture was proved valid by Bohman [2]. We remark that Theorem 1.2 generalizes Bohman's lower bound for the number of edges in $\mathbb{TF}(n)$ and answers a question of Spencer [13]. We discuss in some more details Bohman's result below.

Other results are known for the more general H-free process. In the H-free process, instead of forbidding a triangle, one forbids the appearance of a copy of H. Let $\mathbb{M}(H, n)$ be the graph produced by the H-free process. There are several results with regard to the number of edges in $\mathbb{M}(H, n)$ [2–4, 10, 11, 14]. For a graph $H \neq K_3$ that is strictly 2-balanced, the best lower bounds (which are probably optimal) on the number of edges in $\mathbb{M}(H, n)$ are provided by Bohman and Keevash [3]; the best upper bounds on the number of edges in $\mathbb{M}(H, n)$ are provided by Osthus and Taraz [10] and are within poly$(\ln n)$ factors from the best lower bounds.

Lastly, in [2, 3, 5, 12], the authors consider the independence number of $\mathbb{M}(H, n)$ for some graphs H. Most notable are the results of Bohman [2] and of Bohman and Keevash [3]. Bohman studies the independence number of $\mathbb{M}(H, n)$ for $H \in \{K_3, K_4\}$. His results imply Kim's [8] celebrated lower bound on the off-diagonal Ramsey number $r(3, t)$ and a new lower bound for $r(4, t)$. Bohman and Keevash extend Bohman's results for every H that is strictly 2-balanced. By that, they obtain new lower bounds for the off-diagonal Ramsey numbers $r(s, t)$ for every fixed $s \geq 5$.

1.2 Comparison with Bohman's argument

Bohman's analysis of the triangle-free process in [2] shows that the number of edges in $\mathbb{TF}(n)$ is a.a.s. $\Omega(n^{3/2}\sqrt{\ln n})$. Theorem 1.2 generalizes this result in that it matches Bohman's lower bound up to a constant and in addition provides an a.a.s. lower bound on the number of copies of F in $\mathbb{TF}(n)$, for every fixed F that is a balanced triangle-free graph with density less than 2. Moreover, Bohman's result implies a lower bound of $\Omega(n^{3/2}\sqrt{\ln n})$ on the expected number of edges in $\mathbb{TF}(n)$. Theorem 1.1 generalizes this result in that it matches Bohman's lower bound up to a constant and provides a lower bound on the expected number of copies of F in $\mathbb{TF}(n)$ for every fixed triangle-free graph F. Below we shortly discuss and compare Bohman's argument and ours.

Bohman uses the differential equations method in order to analyse $\mathbb{TF}(n,p)$ for $p = n^{\varepsilon-1/2}$ and some fixed $\varepsilon > 0$. The basic argument can be described as follows. First, a collection of random variables that evolve throughout the random process is introduced and tracked throughout the evolution of $\mathbb{TF}(n,p)$. This collection includes, for example, the random variable $|O_i|$, where O_i denotes the set of edges that have not yet been traversed by the process, and which can be added to the current graph without forming a triangle, after exactly i edges have been added to the evolving graph. Now, at certain times during the process (i.e., at those times in which new edges are added to the evolving graph), the expected change in the values of the random variables in the collection is expressed using the same set of random variables. This allows one to express the random variables in the collection using the solution to an autonomous system of ordinary differential equations. It is then shown that the random variables in the collection are tightly concentrated around the trajectory given by the solution to this system. The particular solution to the system then implies that $|O_I|$ is a.a.s. large for $I = \Omega(n^{3/2}\sqrt{\ln n})$. This then implies the a.a.s. lower bound on the number of edges in $\mathbb{TF}(n)$.

In comparison with the above, we analyse $\mathbb{TF}(n,p)$ for $p = n^{\varepsilon-1/2}$ and some fixed $\varepsilon > 0$ using the original branching process approach of Spencer [12] together with the semi-random method. These two are combined together using combinatorial arguments. Apart from our different approach for the analysis of the triangle-free process, our actual argument is more direct, in the sense that we estimate directly the probability that any fixed triangle-free graph F is included in $\mathbb{TF}(n,p)$. Doing so allows us to infer the validity of our two main results using standard techniques.

We remark that in the course of our analysis, we track and show the concentration of some random variables that in retrospect (and perhaps not surprisingly) turned out to be essentially the same random variables as some of those that were tracked by Bohman. We choose to keep this part of the proof both for the sake of completeness and since it provides an alternative argument for the concentration of these random variables.

Lastly, we note that exactly like Bohman's argument, our ideas can be generalized so as to obtain results which are similar in spirit to our main theorems for the more general H-free process for a large family of graphs H. Moreover, since our arguments allow us to reason about subgraphs other than edges in the evolving graph, we can prove results of the following form: "a.a.s. every set of t vertices in $\mathbb{M}(H,n)$ spans a copy of F" for some t and some fixed graphs F. In particular for $H = K_4$, we can use the ideas presented in this paper in order to show that a.a.s. every set of $t = O(n^{3/5}(\ln n)^{1/5})$ vertices in $\mathbb{M}(K_4,n)$ spans a triangle. This implies an a.a.s. upper bound on the number of edges in $\mathbb{M}(K_4,n)$ which matches up to a constant Bohman's lower bound.

2 Preliminaries

2.1 Notation

As usual, for a natural number a, let $[a] := \{1, 2, \ldots, a\}$. We write $x = a(y \pm z)^b$ if it holds that $x \in [a(y-z)^b, a(y+z)^b]$. We also use $a(y \pm z)^b$ to simply denote the interval $[a(y-z)^b, a(y+z)^b]$. All asymptotic notation in this paper is with respect to $n \to \infty$. All inequalities in this paper are valid only for $n \geq n_0$, for some sufficiently large n_0 which we do not specify.

2.2 Azuma's inequality

The following result is a version of Azuma's inequality [6], tailored for combinatorial applications (see e.g. [7, 9]). Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be independent random variables with α_i taking values in a set A_i. Let $\psi : A_1 \times A_2 \times \ldots \times A_m \to \mathbb{R}$ satisfy the following Lipschitz condition: if two vectors $\alpha, \alpha' \in A_1 \times A_2 \times \ldots \times A_m$ differ only in the ith coordinate, then $|\psi(\alpha) - \psi(\alpha')| \leq c_i$. Then the random variable $X = \psi(\alpha_1, \alpha_2, \ldots, \alpha_m)$ satisfies for any $t \geq 0$,

$$\Pr[|X - \mathbb{E}[X]\,| \geq t] \leq 2\exp\left(-\frac{2t^2}{\sum_{i=1}^m c_i^2}\right).$$

3 Proof of Theorems 1.1 and 1.2

In this section we prove Theorems 1.1 and 1.2, modulo one technical result. We begin by giving an alternative definition of the triangle-free process. Under this alternative definition, we state a result (Theorem 3.1) which trivially implies Theorem 1.1. We then use this result in order to prove Theorem 1.2. The rest of the paper will then be devoted for proving the above mentioned result.

Fix once and for the rest of the paper $\varepsilon \in (0, 10^{-10})$. Define $\delta := 1/\lfloor n^\varepsilon \rfloor$ and $I := \delta^{-2}$. For every integer $i \geq 0$ define a triangle-free graph $\mathbb{TF}_i$ as follows. Initially, take $\mathbb{TF}_0$ to be the empty graph over the vertex set of K_n and set $B_0 := \emptyset$. Given $\mathbb{TF}_i$, define $\mathbb{TF}_{i+1}$ as follows. Choose uniformly at random a function $\beta_{i+1} : K_n \setminus B_{\leq i} \to [0, 1]$ where $B_{\leq i} := \bigcup_{j \leq i} B_j$. Let B_{i+1} be the set of edges f for which the *birthtime* $\beta_{i+1}(f)$ satisfies $\beta_{i+1}(f) < \delta n^{-1/2}$. Traverse the edges in B_{i+1} in order of their birthtimes (starting with the edge whose birthtime is smallest), and add each traversed edge to $\mathbb{TF}_i$, unless its addition creates a triangle. Denote by $\mathbb{TF}_{i+1}$ the graph thus produced. Observe that $\mathbb{TF}_I$ has the same distribution as $\mathbb{TF}(n, p)$ for some $p \sim n^{\varepsilon - 1/2}$.

Let $\Phi(x)$ be a function over the reals, whose derivative is denoted by $\phi(x)$, and which is defined by $\phi(x) := \exp(-\Phi(x)^2)$ and $\Phi(0) := 0$. This is a separable differential equation whose solution (taking into account the initial value) is given implicitly by $\frac{\sqrt{\pi}}{2}\mathrm{erfi}(\Phi(x)) = x$, where $\mathrm{erfi}(x)$ is the imaginary error function, given by $\mathrm{erfi}(x) := \frac{2}{\sqrt{\pi}}\int_0^x \exp(t^2)dt$. We have that $\mathrm{erfi}(x) \to \exp(x^2)/(\sqrt{\pi}x)$ as $x \to \infty$. Hence, it follows that $\Phi(x) \to \sqrt{\ln x}$ as $x \to \infty$.

By the discussion above, linearity of expectation and the fact that the number of copies of F in K_n is $\frac{v_F!}{\mathrm{aut}(F)}\binom{n}{v_F}$ the following result trivially implies Theorem 1.1.

Theorem 3.1. *Let $F \subset K_n$ be a triangle-free graph of size $O(1)$. Then*

$$Pr[F \subseteq \mathbb{TF}_I] \sim \left(\frac{\Phi(I\delta)}{\sqrt{n}}\right)^{e_F}.$$

For a graph F, let Y_F be the random variable that counts the number of copies of F in $\mathbb{TF}_I$. The following theorem clearly implies Theorem 1.2.

Theorem 3.2. *Fix a balanced triangle-free graph F with $e_F/v_F < 2$. Then there exists $0 < \varepsilon_F \leq 10^{-10}$ such that for all $\varepsilon \in (0, \varepsilon_F)$ the following holds. A.a.s.,*

$$Y_F \sim \frac{v_F!}{aut(F)}\binom{n}{v_F}\left(\frac{\Phi(I\delta)}{\sqrt{n}}\right)^{e_F}.$$

Proof. Fix a balanced triangle-free graph F with $e_F/v_F < 2$. Assume $\varepsilon \in (0, \varepsilon_F)$ for some $0 < \varepsilon_F \leq 10^{-10}$ sufficiently small so that it satisfies our arguments below. The number of copies of F in K_n is $\frac{v_F!}{\mathrm{aut}(F)}\binom{n}{v_F}$. Therefore, by Theorem 3.1,

$$\mathbb{E}[Y_F] \sim \frac{v_F!}{\mathrm{aut}(F)}\binom{n}{v_F}\left(\frac{\Phi(I\delta)}{\sqrt{n}}\right)^{e_F}.$$

To complete the proof, it suffices to show that Y_F is concentrated around its mean. For that we use Chebyshev's inequality (see e.g. [1]). Thus it remains to show that $\mathrm{Var}(Y_F) = o(\mathbb{E}[Y_F]^2)$.

For $G \subset K_n$, let I_G be the indicator random variable for the event $\{G \subseteq \mathbb{TF}_I\}$. We have

$$\mathrm{Var}(Y_F) = \sum_{G,G'} \mathrm{Cov}(I_G, I_{G'}) = \sum_{G,G'} \mathbb{E}[I_G I_{G'}] - \mathbb{E}[I_G]\,\mathbb{E}[I_{G'}],$$

where the sum ranges over all copies G, G' of F in K_n. We partition the sum above to two sums and show that each is bounded by $o(\mathbb{E}[Y_F]^2)$. First, let $\sum_{G,G'}$ be the sum over all copies G, G' of F in K_n such that G and G' share no vertex. If G and G' share no vertex then $G \cup G'$ is triangle-free. Hence, since the number of two vertex-disjoint copies of F in K_n is asymptotically equal to the number of copies of F in K_n squared, it follows from Theorem 3.1 that

$$\sum_{G,G'} \mathbb{E}[I_G I_{G'}] - \mathbb{E}[I_G]\,\mathbb{E}[I_{G'}] = o\left(\left(\frac{v_F!}{\mathrm{aut}(F)}\binom{n}{v_F}\left(\frac{\Phi(i\delta)}{\sqrt{n}}\right)^{e_F}\right)^2\right) = o(\mathbb{E}[Y_F]^2).$$

Next, we will make use of the following observation: if G, G' are two copies of F in K_n with $G \cap G'$ being isomorphic to H, then $\mathbb{E}[I_G I_{G'}] = O((n^{\varepsilon-1/2})^{2e_F - e_H})$. This is true since the event $\{G, G' \subseteq \mathbb{TF}_I\}$ implies $\{G \cup G' \subseteq B_{\leq I}\}$ and indeed, $\Pr[G \cup G' \subseteq B_{\leq I}] = O((n^{\varepsilon-1/2})^{2e_F - e_H})$. Let $\sum_H$ be the sum over all $H \subseteq F$ with $v_H \geq 1$. Let $\sum_{G \cap G' \equiv H}$ be the sum over all copies G, G' of F in K_n that share at least 1 vertex such that $G \cap G'$ is isomorphic to H. Then by the observation above,

$$\sum_H \sum_{G \cap G' \equiv H} \mathrm{Cov}(I_G, I_{G'}) \leq O(n^{2v_F - v_H}) \cdot (n^{\varepsilon - 1/2})^{2e_F - e_H},$$

which, since F is a fixed balanced graph with $e_F/v_F < 2$, is at most $o(\mathbb{E}[Y_F]^2)$ if $\varepsilon \in (0, \varepsilon_F)$ and ε_F is sufficiently small. This implies the desired bound on $\mathrm{Var}(Y_F)$. ∎

It remains to prove Theorem 3.1. In the following section we state two technical lemmas that will be used to prove Theorem 3.1. The actual proof of Theorem 3.1 is given in Section 5. The rest of the paper will then be devoted for the proof of these technical lemmas.

4 Technical lemmas

Here we state (and partly prove) two technical lemmas that will be used to prove Theorem 3.1.

We begin with some definitions. For every edge $g \in K_n$ and for every $0 \le i \le I$, $j \in \{0,1,2\}$, define $\Lambda_j(g,i)$ as follows. Let $\Lambda_0(g,i)$ be the family of all sets $\{g_1, g_2\} \subseteq \mathbb{TF}_i$ such that $\{g, g_1, g_2\}$ is a triangle. Let $\Lambda_1(g,i)$ be the family of all singletons $\{g_1\} \subseteq K_n \setminus B_{\le i}$ such that there exists $g_2 \in \mathbb{TF}_i$ for which $\{g, g_1, g_2\}$ is a triangle and it holds that $\mathbb{TF}_i \cup \{g_1\}$ is triangle-free. Let $\Lambda_2(g,i)$ be the family of all sets $\{g_1, g_2\} \subseteq K_n \setminus B_{\le i}$ such that $\{g, g_1, g_2\}$ is a triangle and for which it holds that $\mathbb{TF}_i \cup \{g_j\}$ is triangle-free for both $j \in \{1,2\}$.

Definition 1. *For every $0 \le i \le I$, let*

$$\begin{aligned} \gamma(i) &:= \max\{\delta\Phi(i\delta)\phi(i\delta),\ \delta^2\phi(i\delta)^2\}, \\ \Gamma(i) &:= \begin{cases} n^{-30\varepsilon} & \text{if } i = 0, \\ \Gamma(i-1)\cdot(1+10\gamma(i-1)) & \text{if } i \ge 1. \end{cases} \end{aligned}$$

Our first technical lemma tracks the cardinalities of $\Lambda_j(g,i)$.

Lemma 4.1. *Let $0 \le i < I$. Suppose that given $\mathbb{TF}_i$, we have*

$$\begin{aligned} \forall g \in K_n.\quad |\Lambda_0(g,i)| &\le in^{5\varepsilon}, \\ \forall g \in K_n.\quad |\Lambda_1(g,i)| &\le i\sqrt{n}, \\ \forall g \notin B_{\le i}.\quad |\Lambda_1(g,i)| &= 2\sqrt{n}\Phi(i\delta)\phi(i\delta)\cdot(1\pm\Gamma(i)), \\ \forall g \notin B_{\le i}.\quad |\Lambda_2(g,i)| &= n\phi(i\delta)^2\cdot(1\pm\Gamma(i)). \end{aligned}$$

Then with probability $1 - n^{-\omega(1)}$,

$$\begin{aligned} \forall g \in K_n.\quad |\Lambda_0(g,i+1)| &\le (i+1)n^{5\varepsilon}, \\ \forall g \in K_n.\quad |\Lambda_1(g,i+1)| &\le (i+1)\sqrt{n}, \\ \forall g \notin B_{\le i+1}.\quad |\Lambda_1(g,i+1)| &= 2\sqrt{n}\Phi((i+1)\delta)\phi((i+1)\delta)\cdot(1\pm\Gamma(i+1)), \\ \forall g \notin B_{\le i+1}.\quad |\Lambda_2(g,i+1)| &= n\phi((i+1)\delta)^2\cdot(1\pm\Gamma(i+1)). \end{aligned}$$

The following fact will be used in several places in our proofs, either explicitly or not, and its proof is given in Appendix A.

Fact 4.2. *For all $0 \le i \le I$,*

(i) $1 \ge \phi(i\delta) = \Omega(n^{-1.5\varepsilon})$; $\Phi(i\delta) \le \ln n$; $i \ge 1 \implies \Phi(i\delta) = \Omega(n^{-\varepsilon})$.

(ii) $\gamma(i) = o(1)$; $\gamma(i) = \Omega(n^{-5\varepsilon})$; $n^{-30\varepsilon} \le \Gamma(i) \le n^{-10\varepsilon}$.

4.1 Proof of Lemma 4.1

Fix $0 \le i < I$ and assume that the precondition in Lemma 4.1 holds. We prove that each of the consequences in Lemma 4.1 hold with probability $1 - n^{-\omega(1)}$. Along the way we state a useful lemma that, together with Lemma 4.1, will be used to prove Theorem 3.1 in the next section.

For any $g \in K_n$, assuming $|\Lambda_0(g,i)| \leq in^{5\varepsilon}$, we trivially have that $|\Lambda_0(g,i+1)| \leq in^{5\varepsilon} + \lambda_0(g)$, where $\lambda_0(g)$ is the number of sets $\{g_1\} \in \Lambda_1(g,i)$ for which it holds that $g_1 \in B_{i+1}$, *plus* the number of sets $\{g_1, g_2\} \in \Lambda_2(g,i)$ for which it holds that $g_1, g_2 \in B_{i+1}$. Given the precondition in Lemma 4.1, the fact that $|\Lambda_2(g,i)| \leq n$, the definition of B_{i+1} and the fact that $i < I$, it is clear that $\mathbb{E}[\lambda_0(g)] = o(n^{5\varepsilon})$. Hence, by Chernoff's bound we get that with probability $1 - n^{-\omega(1)}$, $\lambda_0(g) \leq n^{5\varepsilon}$. This implies that with probability $1-n^{-\omega(1)}$, for all $g \in K_n$, $|\Lambda_0(g, i+1)| \leq (i+1)n^{5\varepsilon}$.

Next, for any $g \in K_n$, assuming $|\Lambda_1(g,i)| \leq i\sqrt{n}$, we trivially have that $|\Lambda_1(g,i+1)| \leq i\sqrt{n} + \lambda_1(g)$, where $\lambda_1(g)$ is the number of sets $\{g_1, g_2\} \in \Lambda_2(g,i)$ for which it holds that $g_1 \in B_{i+1}$ and $g_2 \notin B_{i+1}$. By the fact that $|\Lambda_2(g,i)| \leq n$ and by the definition of B_{i+1}, it is clear that $\mathbb{E}[\lambda_1(g)] = o(\sqrt{n})$. Hence, by Chernoff's bound we get that with probability $1-n^{-\omega(1)}$, $\lambda_1(g) \leq \sqrt{n}$. This implies that with probability $1 - n^{-\omega(1)}$, for all $g \in K_n$, $|\Lambda_1(g, i+1)| \leq (i+1)\sqrt{n}$.

Remark 4.3: The only reason we are interested in maintaining the above upper bound on the cardinality of $\Lambda_1(g,i)$ for all $g \in K_n$ and i, is that we need this upper bound in order to maintain an upper bound on the cardinality of $\Lambda_0(g,i)$ for all $g \in K_n$ and i (as we did above). We will not make any further use of the above upper bound on $\Lambda_1(g,i)$.

Having dealt with the easy cases first, we now turn to deal with the two last, more involved consequences in the lemma.

4.1.1 Definitions and an observation

Definition 2 (Redefinition of β_{i+1})**.** *Define $M := n^{20000\varepsilon}$. Let $B^\star_{i+1}$ be a random set of edges, formed by choosing every edge in $K_n \setminus B_{\leq i}$ with probability $Mn^{-1/2}$. For each $g \in B^\star_{i+1}$, let $\beta_{i+1}(g)$ be distributed uniformly at random in $[0, Mn^{-1/2}]$ and for each $g \in K_n \setminus (B_{\leq i} \cup B^\star_{i+1})$, let $\beta_{i+1}(g)$ be distributed uniformly at random in $(Mn^{-1/2}, 1]$.*

Clearly, the above definition of β_{i+1} is equivalent to the original definition of β_{i+1}, given at Section 3. Note that the definition of B_{i+1} is not changed and that $B_{i+1} \subseteq B^\star_{i+1}$.

Let $\Lambda^\star_j(g,i)$ be the family of all $G \in \Lambda_j(g,i)$ such that $G \subseteq B^\star_{i+1}$. Let $\Lambda^{\star\star}_2(g,i)$ be the family of all $G \in \Lambda_2(g,i)$ such that $|G \cap B^\star_{i+1}| = 1$.

Definition 3. *Let $g \in K_n \setminus B_{\leq i}$, $l \in \mathbb{N}$. We define inductively a labeled rooted tree $T^\star_{g,l}$ of height $2l$. The nodes at even distance from the root will be labeled with edges from $K_n \setminus B_{\leq i}$. The nodes at odd distance from the root will be labeled with sets of $j \in \{1,2\}$ edges from $K_n \setminus B_{\leq i}$.*

- $T^\star_{g,1}$:
  - *The root v_0 of $T^\star_{g,1}$ is labeled with the edge g.*
  - *For every $G \in \Lambda^\star_1(g,i) \cup \Lambda^\star_2(g,i)$ do: set a new node u_1, labeled G, as a child of v_0; furthermore, for each edge $g_1 \in G$ set a new node v_1, labeled g_1, as a child of u_1.*
- $T^\star_{g,l}$, $l \geq 2$: *We construct the tree $T^\star_{g,l}$ by adding new nodes to $T^\star_{g,l-1}$ as follows. Let $(v_0, u_1, v_1, \ldots, u_{l-1}, v_{l-1})$ be a directed path in $T^\star_{g,l-1}$ from the root v_0 to a leaf v_{l-1}. Let g_j be the label of v_j.*

- *For every $G \in \Lambda_2^\star(g_{l-1}, i)$ for which $g_{l-2} \notin G$ do: set a new node u_l, labeled G, as a child of v_{l-1}; furthermore, for each edge $g_l \in G$ set a new node v_l, labeled g_l, as a child of u_l.*
- *For every $G \in \Lambda_1^\star(g_{l-1}, i)$ for which $g_{l-2} \notin G$ and $G \cup \{g_{l-1}, g_{l-2}\}$ isn't a triangle do: set a new node u_l, labeled G, as a child of v_{l-1}; furthermore, for the edge $g_l \in G$ set a new node v_l, labeled g_l, as a child of u_l.*

Lastly, for $G \subset K_n \setminus B_{\leq i}$, define $T^\star_{G,l} := \{T^\star_{g,l} : g \in G\}$.

Consider the tree $T^\star_{g,l}$. Let v be a node at even distance from the root of $T^\star_{g,l}$. Let f_0 be the label of v. We define the event that v *survives* as follows. If v is a leaf then v survives by definition. Otherwise, v survives if and only if for every child u, labeled G, of v, the following holds: if $\beta_{i+1}(f) < \min\{\beta_{i+1}(f_0), \delta n^{-1/2}\}$ for all $f \in G$ then u has a child that does not survive. For $g \notin B_{\leq i}$, let $\mathcal{A}_{g,l}$ be the event that the root of $T^\star_{g,l}$ survives. Let $\mathcal{A}_{G,l} := \bigcap_{g \in G} \mathcal{A}_{g,l}$. Given Definition 3, the following is an easy observation.

Proposition 4.4. *Let $l \geq 1$ be an odd integer.*

- *Conditioned on $\{g \in B_{i+1}, \mathbb{TF}_i \cup \{g\}$ is triangle-free$\}$,*

$$\mathcal{A}_{g,l} \implies \{g \in \mathbb{TF}_{i+1}\} \implies \mathcal{A}_{g,l+1}.$$

- *Conditioned on $\{g \notin B_{\leq i+1}, \mathbb{TF}_i \cup \{g\}$ is triangle-free$\}$,*

$$\mathcal{A}_{g,l} \implies \{\mathbb{TF}_{i+1} \cup \{g\} \text{ is triangle-free}\} \implies \mathcal{A}_{g,l+1}.$$

4.1.2 Proof of Lemma 4.1

Let $\mathcal{E}^\star$ be the event that the following properties hold:

P1 For every $g \notin B_{\leq i}$,

$$\begin{aligned}
|\Lambda_1^\star(g,i)| &= 2M\Phi(i\delta)\phi(i\delta) \cdot (1 \pm (\Gamma(i) + o(\Gamma(i)\gamma(i)))), \\
|\Lambda_2^\star(g,i)| &= M^2\phi(i\delta)^2 \cdot (1 \pm (\Gamma(i) + o(\Gamma(i)\gamma(i)))), \\
|\Lambda_2^{\star\star}(g,i)| &= 2M\sqrt{n}\phi(i\delta)^2 \cdot (1 \pm (\Gamma(i) + o(\Gamma(i)\gamma(i)))).
\end{aligned}$$

P2 For every three distinct vertices w, x, y, if $\{w,x\}, \{x,y\} \notin B_{\leq i}$:

- The number of vertices z such that $\{w,z\}, \{y,z\} \in \mathbb{TF}_i$ and $\{x,z\} \in B^\star_{i+1}$ is at most $(\ln n)^2$.
- The number of vertices z such that $\{w,z\} \in \mathbb{TF}_i$ and $\{x,z\}, \{y,z\} \in B^\star_{i+1}$ is at most $(\ln n)^2$.
- The number of vertices z such that $\{w,z\} \in \mathbb{TF}_i$, $\{x,z\} \notin B_{\leq i}$ and $\{y,z\} \in B^\star_{i+1}$ is at most M^2.

P3 For every two distinct vertices x, y, the number of vertices z such that $\{x,z\}, \{y,z\} \in B^\star_{i+1}$ is at most $2M^2$.

P4 For every vertex x, the number of edges $\{x,y\} \in B^\star_{i+1}$ is at most $2M\sqrt{n}$.

Fix once and for the rest of the paper $L \in \{40, 41\}$. The following is our second technical lemma, which is proved in Sections 6–7.

Lemma 4.5.

- *$Pr[\mathcal{E}^\star] = 1 - n^{-\omega(1)}$.*
- *Let $F \subset K_n \setminus B_{\leq i}$ be a triangle-free graph of size $O(1)$ such that $\mathbb{TF}_i \cup F$ is triangle-free. Assume $B^\star_{i+1}$ was chosen and condition on the event that $\mathcal{E}^\star$ holds. Also condition on the event that a_1 edges of F are in B_{i+1} and that a_2 edges of F are not in B_{i+1} (so that $|F| = a_1 + a_2$). Then*

$$Pr[\mathcal{A}_{F,L}] = \left(\frac{\Phi((i+1)\delta) - \Phi(i\delta)}{\phi(i\delta)\,\delta}\right)^{a_1} \left(\frac{\phi((i+1)\delta)}{\phi(i\delta)}\right)^{a_2} \cdot (1 \pm 4\Gamma(i)\gamma(i))^{a_1+a_2}.$$

Corollary 4.6. *Suppose the settings and assumptions in the second item in Lemma 4.5 hold. Further assume that $g \notin B_{\leq i} \cup F$ and condition on $\{g \notin B_{i+1}\}$. Then*

$$Pr[\mathcal{A}_{F,L}] = \left(\frac{\Phi((i+1)\delta) - \Phi(i\delta)}{\phi(i\delta)\,\delta}\right)^{a_1} \left(\frac{\phi((i+1)\delta)}{\phi(i\delta)}\right)^{a_2} \cdot (1 \pm 4.01\Gamma(i)\gamma(i))^{a_1+a_2}.$$

Proof. Suppose the settings and assumptions in the second item in Lemma 4.5 hold and let $g \notin B_{\leq i} \cup F$. Without conditioning on $\{g \notin B_{i+1}\}$, the corollary follows trivially from Lemma 4.5, only with the constant 4.01 being replaced by 4. Now note that we have $\Pr[g \notin B_{i+1}] \geq 1 - \delta M^{-1} \geq 1 - o(\Gamma(i)\gamma(i))$, where the second inequality is by Fact 4.2. This gives the corollary, since given $\mathcal{E}^\star$, $\Pr[\mathcal{A}_{F,L}] = 1 - o(1)$. (Indeed, $\mathcal{A}_{F,L}$ is implied by the event that for all $f \in F$ and for all $G \in \Lambda^\star_j(f,i)$, $j \in \{1,2\}$, there is an edge $g \in G$ which is not in B_{i+1}. Given $\mathcal{E}^\star$ this occurs with probability $1 - o(1)$.) ■

For the rest of the section we assume that we have already made the random choices that determine the set $B^\star_{i+1}$. We also assume that $\mathcal{E}^\star$ holds and keep in mind the fact that this event holds with probability $1 - n^{-\omega(1)}$. We further fix for the rest of the section an edge $g \notin B_{\leq i}$ and condition on the event $\{g \notin B_{i+1}\}$. We estimate the cardinalities of $\Lambda_j(g, i+1)$ for $j \in \{1,2\}$.

We define random variables that will be used to estimate the cardinality of $\Lambda_j(g, i+1)$ for $j \in \{1,2\}$. Let $\lambda_1(g,l)$ be the number of sets $\{g_1\} \in \Lambda_1(g,i)$ for which it holds that $g_1 \notin B_{i+1}$ and $\mathcal{A}_{g_1,l}$ occurs, *plus* the number of sets $\{g_1, g_2\} \in \Lambda^\star_2(g,i) \cup \Lambda^{\star\star}_2(g,i)$ for which it holds that $g_1 \in B_{i+1}$, $g_2 \notin B_{i+1}$, and $\mathcal{A}_{g_1,l} \cap \mathcal{A}_{g_2,l}$ occurs. Let $\lambda_2(g,l)$ be the number of sets $\{g_1, g_2\} \in \Lambda_2(g,i)$ for which it holds that $g_1, g_2 \notin B_{i+1}$ and $\mathcal{A}_{g_1,l} \cap \mathcal{A}_{g_2,l}$ occurs.

By definition of $\Lambda_j(g, i+1)$ and by Proposition 4.4 we have for odd $l \geq 1$,

$$\begin{aligned} \lambda_1(g,l) \;&\leq\; |\Lambda_1(g,i+1)| \;\leq\; \lambda_1(g,l+1), \\ \lambda_2(g,l) \;&\leq\; |\Lambda_2(g,i+1)| \;\leq\; \lambda_2(g,l+1). \end{aligned}$$

Note that since $g \notin B_{\leq i}$, we have for all $F \in \Lambda_1(g,i) \cup \Lambda_2(g,i)$ that $\mathbb{TF}_i \cup F$ is triangle-free. Using this fact, we can use Corollary 4.6 together with the precondition in the lemma, Fact 4.2 and the

fact that $\Pr[f \notin B_{i+1}] \geq 1 - \delta M^{-1}$ to verify that

$$\begin{aligned}\mathbb{E}[\lambda_1(g, L)] &= 2\sqrt{n}\Phi((i+1)\delta)\phi((i+1)\delta) \cdot (1 \pm (\Gamma(i) + 9\Gamma(i)\gamma(i))),\\ \mathbb{E}[\lambda_2(g, L)] &= n\phi((i+1)\delta)^2 \cdot (1 \pm (\Gamma(i) + 9\Gamma(i)\gamma(i))).\end{aligned}$$

We complete the proof by giving concentration results for $\lambda_j(g, L)$, $j \in \{1, 2\}$. The required bound on the cardinality of $\Lambda_j(g, i+1)$ for all $g \notin B_{\leq i+1}$ will then follow from these concentration results, together with a union bound argument.

Concentration of $\lambda_1(g, L)$: Let S_1 be the set of edges which is the union of the sets in $\Lambda_1(g, i)$, $\Lambda_2^\star(g, i)$ and $\Lambda_2^{\star\star}(g, i)$. Let S_2 be the set of all nodes in the trees $T_{f,L}$, $f \in S_1$, where $T_{f,L}$ is defined to be the tree that is obtained as follows: cut off from $T^\star_{f,L}$ every subtree that is rooted at a node having a child that is labeled g. Let $S_3 \supset S_1$ be the set of edges that are labels of nodes in S_2. By the precondition in Lemma 4.1 and $\mathcal{E}^\star$, we have that $|S_1| \leq M^2 n^{1/2}$ and that every tree $T_{f,L}$ has at most $O(M^{2L}) \leq n^{1/1000}$ nodes. Therefore, $|S_3| \leq |S_2| \leq M^2 n^{1/2+1/1000}$. Observe that since we condition on $\{g \notin B_{i+1}\}$, we have that for $f \in S_1$, $\mathcal{A}_{f,L}$ depends only on the birthtimes of the edges that are labels in $T_{f,L}$. Hence, since $S_1 \subseteq S_3$ we have that $\lambda_1(g, L)$ is determined by the birthtimes of the edges in S_3. We argue below that every edge in S_3 appears as a label in at most $n^{1/1000}$ trees $T_{f,L}$, $f \in S_1$. This implies that changing the birthtime of a single edge in S_3 can change $\lambda_1(g, L)$ by at most $n^{1/1000}$. It will then follow from Azuma's inequality, the bound above on the number of edges in S_3, the bound on $\mathbb{E}[\lambda_1(g, L)]$ and Fact 4.2 that, as needed, with probability $1 - n^{-\omega(1)}$,

$$\lambda_1(g, L) = 2\sqrt{n}\Phi((i+1)\delta)\phi((i+1)\delta) \cdot (1 \pm \Gamma(i+1)).$$

We argue that every edge in S_3 appears as a label in at most $n^{1/1000}$ trees $T_{f,L}$, $f \in S_1$. For $g', g'' \in S_3$, say that g' *affects* (resp. *directly-affects*) g'' if there is a tree $T_{f,L}$, $f \in S_1$, with a path (resp. path of length 0 or 2) leading from a node labeled g'' to a node labeled g'. It is enough to show that every edge in S_3 affects at most $n^{1/1000}$ edges in S_1.

Fix $g' \in S_3$. By Definition 3, if $g'' \in S_3 \setminus S_1$ then $g'' \in B^\star_{i+1}$. Therefore, the number of edges $g'' \in S_3 \setminus S_1$ that g' directly-affects is at most $|\Lambda_1^\star(g', i)| + |\Lambda_2^\star(g', i)| + 1 = O(M^2)$, where the upper bound is by $\mathcal{E}^\star$. If g' shares no vertex with g then it is clear that g' directly-affects at most 5 edges in S_1. If g' shares exactly 1 vertex with g then one can verify that given the precondition in Lemma 4.1 and $\mathcal{E}^\star$ (specifically by P1, P2 and P3), g' directly-affects at most $O(M^2)$ edges in S_1. This covers all possible cases since $g' \neq g$ by definition of S_3. We conclude that every edge in S_3 directly-affects $O(M^2)$ edges in S_3. Since a path in $T_{f,L}$ has length at most $2L$, and the edges that are labels along such a path are all in S_3, we get that every edge in S_3 affects $O(M^{2L}) \leq n^{1/1000}$ other edges in S_3. Since $S_1 \subseteq S_3$ we are done.

Remark 4.7: In the argument above, it was essential that we condition on $\{g \notin B_{i+1}\}$. Had we not done that, it would be the case that changing the birthtime of g would change $\lambda_1(g, L)$ potentially by at least $|\Lambda_1(g, i)|$. This affect is too large, as it will render Azuma's inequality useless in providing us with the concentration result we seek.

Concentration of $\lambda_2(g, L)$: Let S_1 be the set of edges which is the union of the sets in $\Lambda_2(g, i)$. Let S_2 be the set of all nodes in the trees $T^\star_{f,L}$, $f \in S_1$. Let $S_3 \supset S_1$ be the set of edges that are labels of nodes in S_2. Trivially, $|S_1| \leq 2n$. Also, by $\mathcal{E}^\star$ every tree $T^\star_{f,L}$ has at most $O(M^{2L}) \leq n^{1/1000}$

nodes. Therefore, $|S_3| \le |S_2| \le 2n^{1+1/1000}$. Observe that $\lambda_2(g, L)$ is determined by the birthtimes of the edges in S_3. We argue below that there is a set of at most $M^2 n^{1/2+1/1000}$ edges in S_3, each of which is a label in at most $M^2 n^{1/2+1/1000}$ trees $T^\star_{f,L}$, $f \in S_1$, and that every other edge in S_3 is a label in at most $n^{1/1000}$ trees $T^\star_{f,L}$, $f \in S_1$. It will then follow from Azuma's inequality, the bound above on the number of edges in S_3, the bound on $\mathbb{E}[\lambda_2(g, L)]$ and Fact 4.2, that as needed, with probability $1 - n^{-\omega(1)}$,

$$\lambda_2(g, L) = n\phi((i+1)\delta)^2 \cdot (1 \pm \Gamma(i+1)).$$

Define *affects* and *directly-affects* exactly as above. It is enough to show that there is a set of at most $M^2 n^{1/2+1/1000}$ edges in S_3, each of which affects at most $M^2 n^{1/2+1/1000}$ edges in S_1, and that every other edge in S_3 affects at most $n^{1/1000}$ edges in S_1.

For a fixed edge $g' \in S_3$, we collect a few useful observations. First assume that $g' \notin B^\star_{i+1}$. Then by Definition 3, we must have that $g' \in S_1$ and that g' appears as a label only at the root of $T^\star_{g',L}$. Therefore, if $g' \notin B^\star_{i+1}$ then g' affects (and directly-affects) only g'. Next assume that $g' \in B^\star_{i+1}$. By $\mathcal{E}^\star$ we have that g' directly-affects at most $|\Lambda^\star_1(g', i)| + |\Lambda^\star_2(g', i)| + 1 = O(M^2)$ edges $g'' \in S_3 \cap B^\star_{i+1} \supset S_3 \setminus S_1$. If g' shares no vertex with g then g' clearly directly-affects at most 5 edges in S_1. If g' shares at least one vertex with g then it follows from the precondition in Lemma 4.1 and $\mathcal{E}^\star$ (specifically by P4) that g' directly-affects at most $M^2 n^{1/2}$ edges in S_1. Lastly we note that for every $f \in S_1$ the following holds: every edge that is a label in $T^\star_{f,L}$, except perhaps for f, is in $S_3 \cap B^\star_{i+1}$.

Say that g' is a *bad-edge* if g' affects an edge $g'' \in S_3 \cap B^\star_{i+1}$ that shares at least one vertex with g. From the observations in the previous paragraph, it follows that if g' is a bad-edge then g' affects at most $M^2 n^{1/2} \cdot O(M^{2L}) \le M^2 n^{1/2+1/1000}$ edges in S_1; on the other hand, if g' is not a bad-edge then g' affects at most $O(M^{2L}) \le n^{1/1000}$ edges in S_1. It thus remains to bound the number of bad-edges. By $\mathcal{E}^\star$ there are at most $M^2 n^{1/2}$ edges $g'' \in B^\star_{i+1}$ that share at least one vertex with g. In addition, by $\mathcal{E}^\star$, for every edge in S_3 there are at most $O(M^{2L}) \le n^{1/1000}$ other edges that affect it. Hence, there are at most $M^2 n^{1/2+1/1000}$ bad-edges. With that we are done.

5 Proof of Theorem 3.1

Let $F \subset K_n$ be a triangle-free graph of size $O(1)$. Say that the triangle-free process *well-behaves* if for every $0 \le i \le I$, the precondition in Lemma 4.1 holds. Note that for $i = 0$ the precondition in Lemma 4.1 holds trivially. Hence, by Lemma 4.1 and the union bound, the process well-behaves with probability $1 - n^{-\omega(1)}$.

For $0 \le i < I$, define

$$\varphi(i) := \frac{\Phi((i+1)\delta) - \Phi(i\delta)}{\delta}.$$

In this section we will use α to denote a *placement* $\{f \in B_{i_f+1} : f \in F\}$, where $0 \le i_f < I$ for all $f \in F$. We will show that for every fixed placement α,

$$\Pr[F \subseteq \mathbb{TF}_I, \text{process well-behaves} \,|\, \alpha] \sim \prod_{f \in F} \varphi(i_f). \tag{1}$$

Note that for every placement α, $\Pr[\alpha] \sim (\frac{\delta}{\sqrt{n}})^{e_F}$. Taking $\sum_\alpha$ to be the sum over all possible placements α, it will then follow from (1) that

$$\begin{aligned}
\Pr[F \subseteq \mathbb{TF}_I, \text{process well-behaves}] &= \sum_\alpha \Pr[\alpha]\Pr[F \subseteq \mathbb{TF}_I, \text{process well-behaves} \,|\, \alpha] \\
&\sim \left(\frac{\delta}{\sqrt{n}}\right)^{e_F} \sum_\alpha \prod_{f\in F} \varphi(i_f) \\
&= \left(\frac{\Phi(I\delta)}{\sqrt{n}}\right)^{e_F},
\end{aligned}$$

where the validity of the last equality is by Claim 5.1 below. Since the process well-behaves with probability $1 - n^{-\omega(1)}$ and $\Phi(I\delta) \to \infty$ as $n \to \infty$, it will then follow that, as needed,

$$\Pr[F \subseteq \mathbb{TF}_I] = n^{-\omega(1)} + \Pr[F \subseteq \mathbb{TF}_I, \text{process well-behaves}] \sim \left(\frac{\Phi(I\delta)}{\sqrt{n}}\right)^{e_F}.$$

Claim 5.1. $\delta^{e_F} \sum_\alpha \prod_{f\in F} \varphi(i_f) = \Phi(I\delta)^{e_F}$.

Proof. Let $\{Z_f : f \in F\}$ be a set of mutually independent 0/1 random variables, defined as follows. For every $f \in F$, choose uniformly at random an index $0 \le i_f < I$ and let $f \in B_{i_f+1}$. Then, let $Z_f = 1$ with probability $\varphi(i_f)$. (We note that $\varphi(i) \in [0, 1]$ for all $0 \le i < I$; see Remark 7.3.) In this context, the probability of a placement α is I^{-e_F}. By definition we have

$$\Pr[\forall f \in F.\ Z_f = 1] = \sum_\alpha \Pr[\alpha]\Pr[\forall f \in F.\ Z_f = 1 \,|\, \alpha] = I^{-e_F} \sum_\alpha \prod_{f\in F} \varphi(i_f).$$

On the other hand, by independence and symmetry we have for every fixed $g \in F$,

$$\Pr[\forall f \in F.\ Z_f = 1]^{1/e_F} = \Pr[Z_g = 1] = I^{-1} \sum_{0\le i_g<I} \varphi(i_g) = (I\delta)^{-1}\Phi(I\delta).$$

■

It remains to prove (1). Fix a placement α. For $0 \le i < I$, define $F_i := F \cap B_{\le i}$. For every $0 \le i \le I$, define the events:

$\mathcal{Q}_1(i)$: The precondition in Lemma 4.1 holds for i.

$\mathcal{Q}_2(i)$: $\mathbb{TF}_i \cup (F \setminus F_i)$ is triangle-free.

$\mathcal{Q}_3(i)$: $F_i \subseteq \mathbb{TF}_i$.

Let $\mathcal{Q}(i) := \mathcal{Q}_1(i) \cap \mathcal{Q}_2(i) \cap \mathcal{Q}_3(i)$. Note that the event $\bigcap_{0\le i\le I} \mathcal{Q}(i)$ is exactly the event $\{F \subseteq \mathbb{TF}_I, \text{process well-behaves} \,|\, \alpha\}$. Therefore, it remains to estimate the probability of $\bigcap_{0\le i\le I} \mathcal{Q}(i)$. Note that $\mathcal{Q}(0)$ holds trivially. The next proposition gives an estimate on the probability that $\mathcal{Q}(i+1)$ holds given $\mathcal{Q}(i)$. Iterating on that proposition for all $0 \le i < I$ gives (1).

Proposition 5.2. *Let $0 \le i < I$ and assume $\mathcal{Q}(i)$ holds. Then $\mathcal{Q}(i+1)$ holds with probability*

$$\left(\frac{\Phi((i+1)\delta) - \Phi(i\delta)}{\phi(i\delta)\,\delta}\right)^{|F_{i+1}\setminus F_i|} \left(\frac{\phi((i+1)\delta)}{\phi(i\delta)}\right)^{|F\setminus F_{i+1}|} \cdot (1 \pm O(n^{-10\varepsilon}))$$

Proof. Assume that we are given an instance of $\mathbb{TF}_i$ and that $\mathcal{Q}(i)$ holds. Consider the process as it creates $\mathbb{TF}_{i+1}$. For the rest of the proof, our context is the one given in Section 4.1.

Since $F \subset K_n$ is triangle-free of size $O(1)$, we have by definition that $F \setminus F_i \subset K_n \setminus B_{\leq i}$ is triangle-free of size $O(1)$. By $\mathcal{Q}(i)$ we have that $\mathbb{TF}_i \cup (F \setminus F_i)$ is triangle-free. Therefore, taking $a_1 = |F_{i+1} \setminus F_i|$ and $a_2 = |F \setminus F_{i+1}|$, it follows from Lemma 4.5 and Fact 4.2 that

$$\Pr[\mathcal{A}_{F\setminus F_i,L}] = \left(\frac{\Phi((i+1)\delta) - \Phi(i\delta)}{\phi(i\delta)\,\delta}\right)^{a_1}\left(\frac{\phi((i+1)\delta)}{\phi(i\delta)}\right)^{a_2} \cdot (1 \pm O(n^{-10\varepsilon})).$$

Let F' be the set of all edges f such that $F \cup \{f\}$ contains a triangle and note that $|F'| = O(1)$. Let $\mathcal{E}'$ be the event that for every $f \in F'$, $\{f \notin \mathbb{TF}_{i+1}\}$. We have that $\mathcal{E}'$ is implied by the event that for every $f \in F'$, $\{f \notin B_{i+1}\}$ occurs. Therefore, $\Pr[\mathcal{E}'] \geq 1 - O(\delta n^{-1/2}) \geq 1 - n^{-10\varepsilon}$. By Lemma 4.1 we have $\Pr[\mathcal{Q}_1(i+1)] = 1 - n^{-\omega(1)}$. Thus, since $\Pr[\mathcal{A}_{F\setminus F_i,L}] = 1 - o(1)$ (see the proof of Corollary 4.6), it follows that

$$\Pr[\mathcal{A}_{F\setminus F_i,L}, \mathcal{E}', \mathcal{Q}_1(i+1)] = \left(\frac{\Phi((i+1)\delta) - \Phi(i\delta)}{\phi(i\delta)\,\delta}\right)^{a_1}\left(\frac{\phi((i+1)\delta)}{\phi(i\delta)}\right)^{a_2} \cdot (1 \pm O(n^{-10\varepsilon})).$$

All that is remained to observe is that the probability of $\{\mathcal{A}_{F\setminus F_i,L}, \mathcal{E}', \mathcal{Q}_1(i+1)\}$ above is an estimation of the probability of $\mathcal{Q}(i+1)$. Indeed, it follows from Proposition 4.4 that if L is odd (resp. even) then $\{\mathcal{A}_{F\setminus F_i,L}, \mathcal{E}', \mathcal{Q}_1(i+1)\}$ implies (resp. is implied by) $\mathcal{Q}(i+1)$. ■

6 Proof of Lemma 4.5

Here we prove Lemma 4.5 modulo one lemma whose proof is given in the next section. Our context in this section and for the rest of the paper is the one given in Section 4.1, where the lemma was stated. That is, we fix $0 \leq i < I$ and assume the precondition in Lemma 4.1 holds.

The first item in Lemma 4.5 follows from Chernoff's bound, using Fact 4.2 and the precondition in Lemma 4.1. Thus it remains to prove the second item in the lemma. For the rest of the paper we assume that $F \subset K_n \setminus B_{\leq i}$ is a triangle-free graph of size $O(1)$ and that the preconditions in the second item of Lemma 4.5 hold. That is, we assume that $\mathbb{TF}_i \cup F$ is triangle-free, $B^\star_{i+1}$ was chosen and $\mathcal{E}^\star$ holds. We also condition on the event that a_1 edges of F are in B_{i+1} and that a_2 edges of F are not in B_{i+1}. We remark that while we do have the set $B^\star_{i+1}$ at hand, we have not yet chosen the random function β_{i+1}.

The basic idea of the proof is as follows. We need to analyse the event $\mathcal{A}_{F,L}$, and the definition of this event calls for a recursive analysis. However, the fact that there could possibly be edges that are labels in more than one node in $T^\star_{F,L}$ makes such a recursive analysis difficult. As we insist on analysing $\mathcal{A}_{F,L}$ recursively, the following observation comes to the rescue. Define $m := \lfloor n^{100\varepsilon} \rfloor \phi(i\delta)^{-1}$. Redefine the birthtime function β_{i+1} as follows. Let B^*_{i+1} be a random set of edges formed by choosing every edge in $B^\star_{i+1}$ with probability mM^{-1}; for each $g \in B^*_{i+1}$, let $\beta_{i+1}(g)$ be distributed uniformly at random in $[0, mn^{-1/2}]$; for each $g \in B^\star_{i+1} \setminus B^*_{i+1}$, let $\beta_{i+1}(g)$ be distributed uniformly at random in $(mn^{-1/2}, Mn^{-1/2}]$ and for all other edges $g \notin B_{\leq i}$, let $\beta_{i+1}(g)$ be distributed uniformly at random in $(Mn^{-1/2}, 1]$. Let $\Lambda^*_j(g,i)$ be the set of all $G \in \Lambda^\star_j(g,i)$ such that $G \subseteq B^*_{i+1}$ and note that $B_{i+1} \subseteq B^*_{i+1} \subseteq B^\star_{i+1}$. Let $T^*_{f,L}$ be defined exactly as $T^\star_{f,L}$ only that now we use in

the definition $\Lambda_j^*(g,i)$ instead of $\Lambda_j^\star(g,i)$. Define $T_{F,L}^* := \{T_{f,L}^* : f \in F\}$. It turns out that with a sufficiently high probability, every edge that is a label in $T_{F,L}^*$ is a label of exactly one node in $T_{F,L}^*$. Moreover, in order to analyse the event $\mathcal{A}_{F,L}$, one only needs to consider the birthtimes of the edges that are labels in $T_{F,L}^*$. This will allow us to analyse $\mathcal{A}_{F,L}$ recursively.

Let $\mathcal{E}^*$ be the event that for every $g \notin B_{\le i}$,

$$\begin{aligned} |\Lambda_1^*(g,i)| &= 2m\Phi(i\delta)\phi(i\delta)\cdot(1 \pm 1.01\Gamma(i)), \\ |\Lambda_2^*(g,i)| &= m^2\phi(i\delta)^2\cdot(1 \pm 1.01\Gamma(i)). \end{aligned}$$

Let $\mathcal{E}_F^*$ be the following event: if g is a label of some node at even distance from the root of a tree in $T_{F,L}^*$, then g is the label of no other node at even distance from the root of a tree in $T_{F,L}^*$.

The following two lemmas correspond to the basic idea outlined above, and clearly imply Lemma 4.5.

Lemma 6.1. *$Pr[\mathcal{E}^*, \mathcal{E}_F^*] = 1 - O(M^{-1/10}) \ge 1 - o(\Gamma(i)\gamma(i))$.*

Lemma 6.2. *Assume B_{i+1}^* was chosen and condition on $\mathcal{E}^* \cap \mathcal{E}_F^*$. Then*

$$Pr[\mathcal{A}_{F,L}] = \left(\frac{\Phi((i+1)\delta) - \Phi(i\delta)}{\phi(i\delta)\,\delta}\right)^{a_1}\left(\frac{\phi((i+1)\delta)}{\phi(i\delta)}\right)^{a_2}\cdot(1 \pm 3.99\Gamma(i)\gamma(i))^{a_1+a_2}.$$

The proof of Lemma 6.1 is given below. The proof of Lemma 6.2 is given in the next section.

6.1 Proof of Lemma 6.1

By Chernoff's bound we have $\Pr[\mathcal{E}^*] = 1 - n^{-\omega(1)}$. Therefore, it is enough to prove that $\Pr[\mathcal{E}_F^*] \ge 1 - M^{-1/10}$. We assume that F is not empty, otherwise the assertion is trivial. For brevity, set $\Lambda^\star(g,i) := \Lambda_1^\star(g,i) \cup \Lambda_2^\star(g,i)$ for all $g \in K_n$. When stating that two graphs share a edges (or vertices), unless otherwise stated this means that the two graphs share exactly a edges (or vertices).

Definition 4 (bad-sequence)**.** *Let $S = (G_1, G_2, \ldots, G_l)$ be a sequence of subgraphs of K_n with $1 \le l \le 2L$. We say that S is a* bad-sequence *if the following properties hold simultaneously.*

- *For every $j \in [l]$: $G_j \in \Lambda^\star(g,i)$ for some $g \in F \cup \bigcup_{k<j} G_k$.*
- *For every $j \in [l-1]$: G_j shares $|G_j|$ vertices and 0 edges with $F \cup \bigcup_{k<j} G_k$.*
- *Either*
  - *G_l shares $|G_l|+1$ vertices and at most $|G_l|-1$ edges with $F \cup \bigcup_{k<l} G_k$, or*
  - *G_l shares $|G_l|$ vertices and 0 edges with $F \cup \bigcup_{k<l} G_k$. In addition, there is an edge $\{x,y\} \in F \cup \bigcup_{k<l} G_k$ such that $G_l \in \Lambda^\star(\{x,y\},i)$ and there is an edge in G_l, without loss of generality $\{x,z\}$, with the following property: there is an edge $\{w,x\} \in F \cup \bigcup_{k<l} G_k$ with $w \ne y$ such that $\{w,z\} \in \mathbb{TF}_i$.*

Let $\mathcal{E}$ be the event that for every bad-sequence $S = (G_1, G_2, \ldots, G_l)$ there exists $j \in [l]$ such that $\{G_j \nsubseteq B_{i+1}^*\}$. The next two propositions imply the desired bound $\Pr[\mathcal{E}_F^*] \ge 1 - M^{-1/10}$, as they state that $\mathcal{E}$ implies $\mathcal{E}_F^*$ and $\Pr[\mathcal{E}] \ge 1 - M^{-1/10}$.

Proposition 6.3. *$\mathcal{E}$ implies $\mathcal{E}_F^*$.*

Proof. Assume $\mathcal{E}$ occurs. We have the following claim.

Claim 6.4. *Let $P = (v_0, u_1, v_1, \dots, u_L, v_L)$ denote an arbitrary path in $T^*_{F,L}$, starting with some root v_0 and ending with some leaf. Let G_j be the label of node u_j and let g_j be the label of node v_j (so that $g_0 \in F$). Then for every $j \in [L]$: G_j shares 0 edges with $F \cup \bigcup_{k<j} G_k$.*

Proof. Suppose for the sake of contradiction that the claim is false, and fix the minimal $l \in [L]$ for which G_l shares at least one edge with $F \cup \bigcup_{k<l} G_k$. Consider the sequence $S = (G_1, G_2, \dots, G_l)$. We shall reach a contradiction by showing that S or some prefix of S is a bad-sequence.

A key observation is this: for all $j \in [l-1]$, G_j shares $|G_j|$ vertices and 0 edges with $F \cup \bigcup_{k<j} G_k$. To see that the observation holds, first note that the minimality of l implies that for all $j \in [l-1]$, G_j shares $0 \leq |G_j| - 1$ edges with $F \cup \bigcup_{k<j} G_k$. In addition, trivially, for all $j \in [l-1]$, G_j shares at least $|G_j|$ vertices with $F \cup \bigcup_{k<j} G_k$. These facts together with $\mathcal{E}$ now imply that there cannot be $j \in [l-1]$ such that G_j shares $|G_j| + 1$ vertices with $F \cup \bigcup_{k<j} G_k$.

Suppose that $|G_l| = 2$. By assumption we have that G_l shares at least one edge with $F \cup \bigcup_{k<l} G_k$, which also implies that G_l shares $|G_l| + 1$ vertices with $F \cup \bigcup_{k<l} G_k$. Hence, by the key observation above, in order to show that S is a bad-sequence and reach a contradiction, it remains to show that G_l shares $1 = |G_l| - 1$ edge with $F \cup \bigcup_{k<l} G_k$. Suppose on the contrary that G_l shares both of its 2 edges with $F \cup \bigcup_{k<l} G_k$. Notice that since F is triangle-free, this implies that $l \geq 2$, so g_{l-2} is well defined. Write $g_{l-2} = \{x, y\}$ and $g_{l-1} = \{x, z\}$ and note that $z \notin \{x, y\}$, $G_{l-1} \in \Lambda^\star(g_{l-2}, i)$ and $G_l \in \Lambda^\star(g_{l-1}, i)$. Now, note that the edge in G_l that is adjacent to z must also be an edge in G_{l-1}. This is true since otherwise, G_{l-1} will share the vertex z with $F \cup \bigcup_{k<l-1} G_k$, which is clearly not the case as by the key observation above G_{l-1} shares only vertices from $\{x, y\}$ with $F \cup \bigcup_{k<l-1} G_k$. The only possible edge to be adjacent in G_l to z and be in G_{l-1} is the edge $\{y, z\}$. Hence we get that y is a vertex of G_l. Therefore, we conclude that $g_{l-2} \in G_l$. But by the definition of $T^*_{F,L}$, $g_{l-2} \notin G_l$. Thus, G_l shares 1 edge with $F \cup \bigcup_{k<l} G_k$ as needed.

Next assume that $|G_l| = 1$. By assumption we have that G_l shares its edge with $F \cup \bigcup_{k<l} G_k$. Since $\mathbb{TF}_i \cup F$ is triangle-free, this implies that $l \geq 2$ and so g_{l-2} is well defined. Let x, y, z be as defined in the previous paragraph. Note that either x or z are vertices of G_l. First we claim that z cannot be a vertex of G_l. Indeed, if z was a vertex of G_l then by a similar argument to that in the previous paragraph we get that G_l must be the edge $\{y, z\}$. But this implies that $\{g_{l-2}, g_{l-1}, g_l\}$ is a triangle and thus contradicts the definition of $T^*_{F,L}$. Therefore, x is a vertex of G_l. We next argue that $(G_j)_{j=1}^{l-1}$ is a bad-sequence, and by that get a contradiction. Note that $\{x, y\}$ is an edge in $F \cup \bigcup_{k<l-1} G_k$ such that $G_{l-1} \in \Lambda^\star(\{x, y\}, i)$ and that $\{x, z\}$ is an edge in G_{l-1}. Let $\{w, x\}$ be the edge in G_l. By definition of $T^*_{F,L}$ we have that $w \notin \{x, y, z\}$. This implies, since we assume that $\{w, x\}$ is an edge in $F \cup \bigcup_{k<l} G_k$, that $\{w, x\}$ is an edge in $F \cup \bigcup_{k<l-1} G_k$. Since $|G_l| = 1$ we have that $\{w, z\} \in \mathbb{TF}_i$. With the key observation above it now follows by definition that $(G_j)_{j=1}^{l-1}$ is a bad-sequence. ■

The next claim, when combined with Claim 6.4, implies the proposition.

Claim 6.5. *Fix $1 \le l \le L$ and let u be a node at distance $2l-1$ from a root in $T^*_{F,L}$. Fix $1 \le l' \le l$ and let u' be a different node at distance $2l'-1$ from a root in $T^*_{F,L}$. Then the labels of u and u' share 0 edges.*

Proof. The proof is by induction on l. For the base case $l = 1$, let u and u' be two distinct nodes at distance 1 from the roots of $T^*_{F,L}$. Let G and G' be the labels of u and u' respectively. Assume for the sake of contradiction that G and G' share at least one edge. We claim that either (G) or (G, G') is a bad-sequence thus reaching the desired contradiction. To see that this indeed holds, note first that by Claim 6.4, G shares $|G|$ vertices and 0 edges with F and G' shares $|G'|$ vertices and 0 edges with F. Let v and v' be the parents of u and u' respectively. Since G and G' share at least one edge and $u \ne u'$, we have that $v \ne v'$. Therefore G and G' share exactly 1 edge. Now, if $|G'| = 2$ it follows that G' shares $|G'| + 1$ vertices and $|G'| - 1$ edges with $F \cup G$; this implies that (G, G') is a bad-sequence. Next, assume $|G'| = 1$. Let $\{x, y\} \in F$ and $\{w, x\} \in F$ be the labels of v and v' respectively. Let z be the vertex of G and G' that is not in F so that G and G' share the edge $\{x, z\}$. Clearly $w \ne y$. In addition, since $|G'| = 1$ we have that $\{w, z\} \in \mathbb{TF}_i$. It follows that (G) is a bad-sequence.

Fix $2 \le l \le L$ and assume the claim is valid for $l - 1$. Let u be a node at distance $2l - 1$ from a root in $T^*_{F,L}$. Fix $1 \le l' \le l$ and let u' be a different node at distance $2l' - 1$ from a root in $T^*_{F,L}$. Assume for the sake of contradiction that the label of u shares at least one edge with the label of u'. Without loss of generality we further assume that l' is minimal in the following sense: the label of u shares 0 edges with the label of every node at odd distance less than $2l' - 1$ from the root of $T^*_{F,L}$. By Claim 6.4, we may also assume that u' is not a node on the path from a root to u in $T^*_{F,L}$.

Let P be the unique path from a root to u in $T^*_{F,L}$. Let P' be the longest unique path in $T^*_{F,L}$ that ends with u' and which do not contain a node from P. Traverse the nodes along the path P and then traverse the nodes along the path P', ending each traversal at the nodes u and u' respectively. Let $(u_1, u_2, \ldots, u_s)$ be the nodes so traversed that are at odd distances from the roots of the forest, in order of their traversal. By construction, $u_l = u$ and $u_s = u'$. We note that $2 \le s \le 2L$. Let G_j be the label of node u_j and set $S_1 = (G_1, G_2, \ldots, G_s)$. Let $S_2 = (G_1, G_2, \ldots, G_{l-1}, G_{l+1}, G_{l+2}, \ldots, G_{s-1}, G_l)$. In words, S_2 is obtained from S_1 by first removing G_l and G_s and then concatenating G_l to the end of the new sequence. Note that G_l and G_s are the labels of u and u' respectively and that by assumption G_l and G_s share at least one edge. We show below that either S_1 or S_2 is a bad-sequence and by that reach the desired contradiction.

Assume that $|G_s| = 2$. We show that S_1 is a bad-sequence. By Claim 6.4, the minimality of l' and the induction hypothesis we have that for every $j \in [s-1]$, G_j shares 0 edges with $F \cup \bigcup_{k<j} G_k$. Therefore, by $\mathcal{E}$ we also have that for every $j \in [s-1]$, G_j shares $|G_j|$ vertices with $F \cup \bigcup_{k<j} G_k$. Let v_l be the parent of u_l and v_s the parent of u_s. Let g_l be the label of v_l and g_s the label of v_s. Since $u_l \ne u_s$ and yet G_l and G_s share at least one edge, we get that $v_l \ne v_s$. This implies by Claim 6.4 and the induction hypothesis that $g_l \ne g_s$. This, in turn, implies that G_l shares *exactly* 1 edge with G_s. In what follows we show that G_s shares 0 edges with $F \cup \bigcup_{k<l, l<k<s} G_k$. This will give us that G_s shares $|G_s| + 1$ vertices and $1 = |G_s| - 1$ edges with $F \cup \bigcup_{k<s} G_k$, which given the above implies that S_1 is a bad-sequence. The fact that G_s shares 0 edges with $F \cup \bigcup_{l<k<s} G_k$ follows from Claim 6.4. We claim that G_s shares 0 edges with $\bigcup_{k<l} G_k$. Indeed, if G_s does share at least one edge with $\bigcup_{k<l} G_k$, then since G_s also shares at least one edge with G_l, we get that

G_l shares $|G_l|+1$ vertices with $F \cup \bigcup_{k<l} G_k$. But since $l \in [s-1]$, we've ruled out that possibility above.

Assume that $|G_s| = 1$. We show that S_2 is a bad-sequence. For brevity, rewrite $S_2 = (F_1, F_2, \ldots, F_{s-1})$ and note that F_{s-1} is the label of u_l. By Claim 6.4, the minimality of l' and the induction hypothesis we have that for every $j \in [s-1]$, F_j shares 0 edges with $F \cup \bigcup_{k<j} F_k$. Therefore, by $\mathcal{E}$ we also have that for every $j \in [s-1]$, F_j shares $|F_j|$ vertices with $F \cup \bigcup_{k<j} F_k$. Define g_l, g_s as in the previous paragraph and note that for the same reasons as above we have that $g_l \neq g_s$. Also note that $g_l, g_s \in F \cup \bigcup_{k<s-1} F_k$. Write $g_l = \{x, y\}$ and let z be the vertex of G_l that is not in $\{x, y\}$. Assume without loss of generality that G_l and G_s share the edge $\{x, z\}$. Since z is not a vertex of $F \cup \bigcup_{k<s-1}$, we get that x is a vertex in g_s. Write $g_s = \{w, x\}$ and note that $w \neq y$. Lastly, since $|G_s| = 1$ we have that $\{w, z\} \in \mathbb{TF}_i$. Therefore, by definition, S_2 is a bad-sequence. ■

With that we complete the proof of the proposition. ■

Proposition 6.6. *$Pr[\mathcal{E}] \geq 1 - M^{-1/10}$.*

Proof. For a bad-sequence $S = (G_1, G_2, \ldots, G_l)$, write $\{S \subseteq B^*_{i+1}\}$ for the event that for all $j \in [l]$, $\{G_j \subseteq B^*_{i+1}\}$. Let Z be the random variable that counts the number of bad-sequences S for which $\{S \subseteq B^*_{i+1}\}$. It suffices to show that $\mathbb{E}[Z] \leq M^{-1/10}$.

For $l \in [2L]$, $0 \leq c < l$, let $\mathrm{Seq}_1(l, c)$ denote the set of all bad-sequences $S = (G_1, G_2, \ldots, G_l)$ with $c = |\{j : |G_j| = 1, j < l\}|$ such that G_l shares $|G_l|+1$ vertices and at most $|G_l|-1$ edges with $F \cup \bigcup_{k<l} G_k$. For $l \in [2L]$, $0 \leq c < l$, let $\mathrm{Seq}_2(l, c)$ denote the set of all bad-sequences $S = (G_1, G_2, \ldots, G_l)$ with $c = |\{j : |G_j| = 1, j < l\}|$ that are not in $\mathrm{Seq}_1(l, c)$. Then

$$\mathbb{E}[Z] \quad = \quad \sum_{l \in [2L]} \sum_{0 \leq c < l} \sum_{j \in \{1,2\}} \sum_{S \in \mathrm{Seq}_j(l,c)} \Pr[S \subseteq B^*_{i+1}]. \tag{2}$$

Below we show that

$$\forall l \in [2L],\ 0 \leq c < l. \quad \sum_{S \in \mathrm{Seq}_1(l,c)} \Pr[S \subseteq B^*_{i+1}] \quad \leq \quad M^{-1/9}, \tag{3}$$

$$\forall l \in [2L],\ 0 \leq c < l. \quad \sum_{S \in \mathrm{Seq}_2(l,c)} \Pr[S \subseteq B^*_{i+1}] \quad \leq \quad M^{-1/9}. \tag{4}$$

From (2), (3) and (4) and since $L = O(1)$, we get that $\mathbb{E}[Z] \leq M^{-1/10}$ as required.

We prove (3). Fix $l \in [2L]$, $0 \leq c < l$. We first count the number of sequences $S = (G_1, G_2, \ldots, G_l)$ in $\mathrm{Seq}_1(l, c)$. To do so, we construct such a sequence iteratively. First, we choose the cardinalities of the first $l-1$ subgraphs in S. Note that there are $\binom{l-1}{c} = O(1)$ possible choices for the cardinalities. Suppose we have already chosen the first $j-1$ subgraphs in S for some $j < l$. Given that, we count the number of choices for G_j assuming $j \geq 1$. There are $O(1)$ possible choices for an edge $g \in F \cup \bigcup_{k<j} G_k$ for which $G_j \in \Lambda^\star(g, i)$. Given g: if $|G_j|$ is to be of size 1 then there are at most $\Lambda^\star_1(g, i)$ choices for G_j and if $|G_j|$ is to be of size 2 then there are at most $\Lambda^\star_2(g, i)$ choices for G_j. Given that we have already chosen the first $l-1$ subgraphs in S, the number of

choices for G_l is at most $O(1)$, since the vertices of G_l are all in $F \cup \bigcup_{k<l} G_k$. Therefore, by $\mathcal{E}^\star$ the number of sequences in $\mathrm{Seq}_1(l, c)$ is at most

$$O(1) \cdot \left(M^2 \phi(i\delta)^2\right)^{l-1-c} \cdot \left(M \Phi(i\delta) \phi(i\delta)\right)^c.$$

Even if we condition on the event that a_1 edges of F are in B_{i+1} and a_2 edges of F are not in B_{i+1}, we get that the probability of $\{S \subseteq B^*_{i+1}\}$ for $S \in \mathrm{Seq}_1(l, c)$ is at most

$$\left(\frac{m^2}{M^2}\right)^{l-1-c} \cdot \left(\frac{m}{M}\right)^c \cdot \frac{m}{M}. \tag{5}$$

Hence,

$$\begin{aligned}
\sum_{S \in \mathrm{Seq}_1(l,c)} \Pr[S \subseteq B^*_{i+1}] &\leq O(1) \cdot \left(m^2 \phi(i\delta)^2\right)^{l-1-c} \cdot \left(m \Phi(i\delta) \phi(i\delta)\right)^c \cdot \frac{m}{M} \\
&\leq O(1) \cdot m^{2l-2-2c} \cdot (m \ln n)^c \cdot \frac{m}{M} \\
&\leq M^{-1/9},
\end{aligned}$$

where the second inequality follows from Fact 4.2 and the last inequality follows from the definition of L, m and M. This gives us the validity of (3).

It remains to prove (4). Fix $l \in [2L]$, $0 \leq c < l$. As before, we first count the number of sequences $S = (G_1, G_2, \dots, G_l)$ in $\mathrm{Seq}_2(l, c)$ and we do it by constructing such a sequence iteratively. The number of choices for the first $l-1$ subgraphs in S is exactly as in the previous case. Suppose we have already chosen the first $l-1$ subgraphs in S. We claim that the number of choices for G_l is at most $O((\ln n)^2)$. Indeed, there are $O(1)$ choices for an edge $\{x, y\} \in F \cup \bigcup_{k<l} G_k$ such that $G_l \in \Lambda^\star(\{x, y\}, i)$. Given $\{x, y\}$, there are at most $O(1)$ choices for an edge $\{w, x\} \in F \cup \bigcup_{k<l} G_k$ such that $w \neq y$. Furthermore, given $\{x, y\}$ and $\{w, x\}$, by $\mathcal{E}^\star$ (specifically by P2) there are at most $2(\ln n)^2$ choices for $G_l \in \Lambda^\star(\{x, y\}, i)$ which has a vertex z that is not a vertex of $F \cup \bigcup_{k<l} G_k$ and such that $\{x, z\} \in G_l$ and $\{w, z\} \in \mathbb{TF}_i$. Therefore, by $\mathcal{E}^\star$ the number of sequences in $\mathrm{Seq}_2(l, c)$ is at most

$$O(1) \cdot \left(M^2 \phi(i\delta)^2\right)^{l-1-c} \cdot \left(M \Phi(i\delta) \phi(i\delta)\right)^c \cdot (\ln n)^2.$$

Even if we condition on the event that a_1 edges of F are in B_{i+1} and a_2 edges of F are not in B_{i+1}, we get that the probability of $\{S \subseteq B^*_{i+1}\}$ for $S \in \mathrm{Seq}_2(l, c)$ is at most as given in (5). Therefore,

$$\begin{aligned}
\sum_{S \in \mathrm{Seq}_2(l,c)} \Pr[S \subseteq B^*_{i+1}] &\leq O(1) \cdot \left(m^2 \phi(i\delta)^2\right)^{l-1-c} \cdot \left(m \Phi(i\delta) \phi(i\delta)\right)^c \cdot \frac{m}{M} \cdot (\ln n)^2 \\
&\leq O(1) \cdot m^{2l-2-2c} \cdot (m \ln n)^c \cdot \frac{m}{M} \cdot (\ln n)^2 \\
&\leq M^{-1/9},
\end{aligned}$$

where as before, the second inequality follows from Fact 4.2 and the last inequality follows from the definition of L, m and M. This gives us the validity of (4). With that we complete the proof. ■

7 Proof of Lemma 6.2

Assume that B^*_{i+1} was chosen and condition on $\mathcal{E}^* \cap \mathcal{E}^*_F$. Fix an edge $f \in F$. Note that we either condition on the event that f is in B_{i+1} or not. For simplicity of presentation, we do not choose right now which of these two options hold. The exact choice will be made implicitly below, whenever we condition on an event which is concerned with the birthtime $\beta_{i+1}(f)$.

Some remarks regarding $T^*_{f,L}$ follow. The event $\mathcal{E}^*_F$ says that every label of some node in $T^*_{f,L}$ is a label of exactly one node in $T^*_{f,L}$. Therefore, we shall refer from now on to the nodes of $T^*_{f,L}$ *by their labels*. The event $\mathcal{E}^*$ implies, using the definition of $T^*_{f,L}$ and Fact 4.2, that for every non-leaf node g at even distance from the root of $T^*_{f,L}$,

$$\begin{aligned}\text{Number of children of } g \text{ that are of size 1} &= 2m\Phi(i\delta)\phi(i\delta)(1 \pm 1.02\Gamma(i)), &(6)\\ \text{Number of children of } g \text{ that are of size 2} &= m^2\phi(i\delta)^2(1 \pm 1.02\Gamma(i)). &(7)\end{aligned}$$

We need to define the following two additional rooted trees.

Definition 5 (T_∞, T_l)**.**

- *Let T_∞ be an infinite rooted tree, defined as follows. Every node g at even distance from the root has two sets of children. One set consists of children which are singletons and the other set consists of children which are sets of size 2. Every node G at odd distance from the root of T_∞, which is a set of size $|G| \in \{1,2\}$, has exactly $|G|$ children. Lastly, for every node g at even distance from the root:*

$$\begin{aligned}\textit{Number of children of } g \textit{ that are of size } 1 &= \lceil 2m\Phi(i\delta)\phi(i\delta)\rceil,\\ \textit{Number of children of } g \textit{ that are of size } 2 &= m^2\phi(i\delta)^2.\end{aligned}$$

- *Let $0 \le l \le L$. Define T_l to be the tree that is obtained by cutting from T_∞ every subtree that is rooted at a node whose distance from the root of T_∞ is larger than $2l$.*

Remark 7.1: Note that $m^2\phi(i\delta)^2$ is an integer. It would be convenient to assume from now on that $2m\Phi(i\delta)\phi(i\delta)$ is also an integer. Hence, for example, the number of children of the root of T_∞ that are of size 1 is exactly $2m\Phi(i\delta)\phi(i\delta)$. We explain in Section 7.4 how to modify our proof for the case where $2m\Phi(i\delta)\phi(i\delta)$ is not an integer.

We continue with some more setup. Note that for every node $g \neq f$ at even distance from the root of $T^*_{f,L}$, $\beta_{i+1}(g)$ is distributed uniformly at random in the interval $[0, mn^{-1/2}]$. We extend the definition of β_{i+1} so that in addition, for every node g at even distance from the root of T_∞ (and hence from the root of T_L), the *birthtime* $\beta_{i+1}(g)$ is distributed uniformly at random in the interval $[0, mn^{-1/2}]$.

Let $T \in \{T^*_{f,L}, T_L, T_\infty\}$. Let g_0 be a node at even distance from the root of T. We define the event that g_0 *survives* as follows. If g_0 is a leaf (so that $T \neq T_\infty$) then g_0 survives by definition. Otherwise, g_0 survives if and only if for every child G of g_0, the following holds: if $\beta_{i+1}(g) < \min\{\beta_{i+1}(g_0), \delta n^{-1/2}\}$ for all children g of G, then G has a child that does not survive.

For a node g at height $2l$ in $T^*_{f,L}$, let $p_{g,l}(x)$ be the probability that g survives under the assumption that $\beta_{i+1}(g) = xn^{-1/2}$. Let $p_l(x)$ be the probability, at the limit as $n \to \infty$, that the root of T_l survives under the assumption that $\beta_{i+1}(g) = xn^{-1/2}$, where g here denotes the root of T_l. Let $p(x)$ be the probability, at the limit as $n \to \infty$, that the root of T_∞ survives under the assumption that $\beta_{i+1}(g) = xn^{-1/2}$, where g here denotes the root of T_∞. One can show that $p_{g,l}(x), p_l(x)$ and $p(x)$ are all continuous and bounded in the interval $[0, \delta]$. Hence, we can define the following functions on the interval $[0, \delta]$:

$$P_{g,l}(x) := \int_0^x p_{g,l}(y)dy, \qquad P_l(x) := \int_0^x p_l(y)dy \quad \text{and} \quad P(x) := \int_0^x p(y)dy.$$

Observe that for all $x \in (0, \delta]$:

$$\begin{aligned}
\Pr[\text{The root } f \text{ of } T^*_{f,L} \text{ survives} \,|\, \beta_{i+1}(f) < xn^{-1/2}] &= \frac{P_{f,L}(x)}{x}, \\
\lim_{n\to\infty} \Pr[\text{The root } g \text{ of } T_l \text{ survives} \,|\, \beta_{i+1}(g) < xn^{-1/2}] &= \frac{P_l(x)}{x}, \\
\lim_{n\to\infty} \Pr[\text{The root } g \text{ of } T_\infty \text{ survives} \,|\, \beta_{i+1}(g) < xn^{-1/2}] &= \frac{P(x)}{x}.
\end{aligned}$$

The next lemma, when combined with the discussion above and the definition of $\mathcal{A}_{F,L}$, implies Lemma 6.2.

Lemma 7.2.

(i) $P(\delta) = \frac{\Phi((i+1)\delta) - \Phi(i\delta)}{\phi(i\delta)}$ *and* $p(\delta) = \frac{\phi((i+1)\delta)}{\phi(i\delta)}$.

(ii) For all $x \in [0, \delta]$, $p_L(x) = p(x)(1 \pm o(\Gamma(i)\gamma(i)))$.

(iii) For all $x \in [0, \delta]$, $p_{f,L}(x) = p_L(x)(1 \pm 3\Gamma(i)\gamma(i))$.

The proof of Lemma 7.2 is given in the next three subsections.

7.1 Proof of Lemma 7.2 (i)

Clearly $p(0) = 1$ and $P(0) = 0$. Hence, from the definition of survival and the definition of $p(x)$ and $P(x)$, we get that for every $x \in [0, \delta]$, at the limit as $n \to \infty$,

$$\begin{aligned}
p(x) &= \left(1 - \frac{P(x)^2}{m^2}\right)^{m^2\phi(i\delta)^2} \left(1 - \frac{P(x)}{m}\right)^{2m\Phi(i\delta)\phi(i\delta)} \\
&= \exp\Big(- P(x)^2\phi(i\delta)^2 - 2P(x)\Phi(i\delta)\phi(i\delta)\Big).
\end{aligned} \tag{8}$$

By the fundamental theorem of calculus, $p(x)$ is the derivative of $P(x)$. Hence, we view (8) as the separable differential equation that it is. This equation has the following as an implicit solution:

$$\int \exp\left(P^2\phi(i\delta)^2 + 2P\phi(i\delta)\Phi(i\delta)\right) dP = x.$$

Solving the above integral, we get

$$\frac{\sqrt{\pi}}{2}\operatorname{erfi}\big(\Phi(i\delta)+\phi(i\delta)P\big)=x+C. \tag{9}$$

With the initial condition $P(0)=0$, we get from (9) that

$$\frac{\sqrt{\pi}}{2}\operatorname{erfi}(\Phi(i\delta))=C.$$

Let $z\ge 0$ satisfy

$$\exp(-z^2\phi(i\delta)^2-2z\phi(i\delta)\Phi(i\delta))=\frac{\phi((i+1)\delta)}{\phi(i\delta)}.$$

A simple analysis shows that

$$z=\frac{\Phi((i+1)\delta)-\Phi(i\delta)}{\phi(i\delta)}.$$

Taking $P=z$ and $C=\frac{\sqrt{\pi}}{2}\operatorname{erfi}(\Phi(i\delta))$, we solve (9) for x to get

$$x=\frac{\sqrt{\pi}}{2}\operatorname{erfi}\big(\Phi(i\delta)+\phi(i\delta)P\big)-C=\frac{\sqrt{\pi}}{2}\big(\operatorname{erfi}(\Phi((i+1)\delta))-\operatorname{erfi}(\Phi(i\delta))\big)=\delta,$$

where the last equality is by the fact that $\frac{\sqrt{\pi}}{2}\operatorname{erfi}(\Phi(x))=x$. Hence, $P(\delta)=\frac{\Phi((i+1)\delta)-\Phi(i\delta)}{\phi(i\delta)}$ and $p(\delta)=\frac{\phi((i+1)\delta)}{\phi(i\delta)}$. This completes the proof.

Remark 7.3: As a side note, we observe that $0\le P(\delta)\le\delta$. Hence we get from the above conclusion and from Fact 4.2 that $\frac{\Phi((i+1)\delta)-\Phi(i\delta)}{\delta}=P(\delta)\phi(i\delta)/\delta\ge 0$ and that $\frac{\Phi((i+1)\delta)-\Phi(i\delta)}{\delta}=P(\delta)\phi(i\delta)/\delta\le 1$.

7.2 Proof of Lemma 7.2 (ii)

Assume first that L is odd. Let g_0 be the root of T_L and T_∞. Further assume $\beta_{i+1}(g_0)=xn^{-1/2}$ for some $x\in[0,\delta]$. Clearly if g_0 survives in T_L then g_0 survives in T_∞. Hence $p_L(x)\le p(x)$. Below we show that $p_L(x)\ge p(x)-n^{-36\varepsilon}$. We claim that this last inequality implies $p_L(x)=p(x)(1-o(\Gamma(i)\gamma(i)))$, which gives the lemma. Indeed, using the fact that $x\le\delta$ and since trivially $P(x)\le x$, it follows from (8), the definition of δ and Fact 4.2 that $p(x)\sim 1$. In addition, by Fact 4.2 we have that $\Gamma(i)\gamma(i)=\Omega(n^{-35\varepsilon})$. Therefore we get, as needed,

$$p_L(x)\ge p(x)(1-n^{-36\varepsilon}/p(x))=p(x)(1-o(\Gamma(i)\gamma(i))).$$

Say that a node g at even distance from the root of T_L is *relevant*, if g and its sibling (if exists) have a smaller birthtime than their grandparent, and in addition, their grandparent is either relevant or the root. Observe that if the root of T_∞ survives then either the root of T_L survives, or else, there is a relevant leaf in T_L. It remains to show that the expected number of relevant leaves in T_L is at most $n^{-36\varepsilon}$.

Say that a leaf g_L in T_L is a *c-type* if the path leading from the root to g_L contains exactly c nodes G at odd distance from the root, which are sets of size 1. Consider a path $(g_0,G_1,g_1,\ldots,G_L,g_L)$

from the root to a leaf g_L, where g_L is a c-type. Let $\mathcal{G}$ be the union of $\{g_j : j \in [L]\}$ together with the set $\{g : g \text{ is a sibling of some } g_j,\ j \in [L]\}$. Since g_L is a c-type, we have $|\mathcal{G}| = 2L - c$. Now if g_L is relevant, then for every node $g \in \mathcal{G}$, $\{\beta_{i+1}(g) < \beta_{i+1}(g_0) = xn^{-1/2}\}$ holds. This event occurs with probability $(x/m)^{2L-c}$. Hence, the probability that g_L is relevant is at most

$$\left(\frac{x}{m}\right)^{2L-c} = \left(\frac{x}{m}\right)^{c}\left(\frac{x^2}{m^2}\right)^{L-c}.$$

The number of c-type leaves in T_L is at most

$$2^L\left(2m\Phi(i\delta)\phi(i\delta)\right)^c\left(2m^2\phi(i\delta)^2\right)^{L-c} \le (4m\ln n)^c\,(4m^2)^{L-c},$$

where the inequality is by Fact 4.2. Hence, the expected number of relevant c-type leaves in T_L is at most

$$\left(\frac{x}{m}\right)^{c}\left(\frac{x^2}{m^2}\right)^{L-c}(4m\ln n)^c\,(4m^2)^{L-c} \le (4x\ln n)^{2L-c}.$$

Now, $(4x\ln n)^{2L-c} \le \delta^{2L-c}(4\ln n)^{2L-c} \le \delta^{L-1} \sim n^{-40\varepsilon}$, where the inequalities are by $x \le \delta$, $c \le L$ and $(4\ln n)^{2L} \le \delta^{-1}$. To complete the proof, note that if a leaf is a c-type, then we have at most $L+1 = O(1)$ possible choices for c. Therefore, with the union bound we conclude that the expected number of relevant leaves in T_L is at most $n^{-36\varepsilon}$.

Next assume that L is even, let g_0 be as above and assume $\beta_{i+1}(g_0) = xn^{-1/2}$. The proof for this case is similar to the previous case and so we only outline it. It is easy to verify that if g_0 doesn't survive in T_L then g_0 doesn't survive in T_∞. Hence $p_L(x) \ge p(x)$. Now, if g_0 doesn't survive in T_∞ then either the root of T_L doesn't survive, or else, there is a relevant leaf in T_L. One can now show using the same argument as above that the expected number of relevant leaves in T_L is at most $n^{-36\varepsilon}$. This completes the proof.

7.3 Proof of Lemma 7.2 (iii)

The following implies Lemma 7.2 (iii).

Proposition 7.4. *Let $x \in [0,\delta]$, $0 \le l \le L$. Let g be a node at height $2l$ in $T^*_{f,L}$. Then*

$$p_{g,l}(x) = p_l(x)(1 \pm 3\Gamma(i)\gamma(i)).$$

Proof. The proof is by induction on l. The assertion holds for the base case since by definition, $p_{g,0}(x) = p_0(x) = 1$ for all $x \in [0,\delta]$. Let $1 \le l \le L$ and assume that the proposition holds for $l-1$. Fix $x \in [0,\delta]$ and let g be a node at height $2l$ in $T^*_{f,L}$.

For brevity, define $\eta := \Gamma(i)\gamma(i)$. Further, let

$$Q^* := \begin{array}{l}\left(1 - \dfrac{P_{l-1}(x)(1-3\eta)}{m}\right)^{2m\Phi(i\delta)\phi(i\delta)(1-1.02\Gamma(i))} \\ \left(1 - \dfrac{P_{l-1}(x)^2(1-3\eta)^2}{m^2}\right)^{m^2\phi(i\delta)^2(1-1.02\Gamma(i))}\end{array}.$$

and

$$Q_* := \quad \Big(1 - \frac{P_{l-1}(x)(1+3\eta)}{m}\Big)^{2m\Phi(i\delta)\phi(i\delta)(1+1.02\Gamma(i))} \cdot$$
$$\Big(1 - \frac{P_{l-1}(x)^2(1+3\eta)^2}{m^2}\Big)^{m^2\phi(i\delta)^2(1+1.02\Gamma(i))}.$$

Let g' be a grandchild of g. By the induction hypothesis and by definition of $P_{g',l-1}(x)$ and $P_{l-1}(x)$,

$$P_{g',l-1}(x) = P_{l-1}(x)(1 \pm 3\eta).$$

Thus, it follows from the definition of survival and by (6) and (7) that

$$Q_* \leq p_{g,l}(x) \leq Q^*.$$

It remains to bound Q^* and Q_*. In what follows we use the fact that

$$\forall z > 1. \;\; \exp(-1/(z-1)) < 1 - 1/z < \exp(-1/z). \tag{10}$$

To bound Q^*, we have

$$\begin{aligned}
\Big(1 - \frac{P_{l-1}(x)(1-3\eta)}{m}\Big)^{2m\Phi(i\delta)\phi(i\delta)} &\leq \Big(1 - \frac{P_{l-1}(x)}{m}\Big)^{2m\Phi(i\delta)\phi(i\delta)(1-O(\eta))(1-1/m)} \\
&\leq \Big(1 - \frac{P_{l-1}(x)}{m}\Big)^{2m\Phi(i\delta)\phi(i\delta)(1-O(\eta))} \\
&\leq \Big(1 - \frac{P_{l-1}(x)}{m}\Big)^{2m\Phi(i\delta)\phi(i\delta)} \Big(1 - \frac{\delta}{m}\Big)^{-O(m\Phi(i\delta)\phi(i\delta)\eta)} \\
&\leq \exp\big(-2P_{l-1}(x)\Phi(i\delta)\phi(i\delta)\big) \cdot (1+o(\eta)),
\end{aligned}$$

where the first inequality follows from (10); the second inequality follows from the fact that $1/m = o(\eta)$, which in turn follows from the definition of m and from Fact 4.2; the third inequality follows since $P_{l-1}(x) \leq x \leq \delta$; and the last inequality follows from (10) and Fact 4.2. For similar reasons we also have that

$$\begin{aligned}
\Big(1 - \frac{P_{l-1}(x)^2(1-3\eta)^2}{m^2}\Big)^{m^2\phi(i\delta)^2} &\leq \Big(1 - \frac{P_{l-1}(x)^2}{m^2}\Big)^{m^2\phi(i\delta)^2(1-O(\eta))(1-1/m)} \\
&\leq \Big(1 - \frac{P_{l-1}(x)^2}{m^2}\Big)^{m^2\phi(i\delta)^2(1-O(\eta))} \\
&\leq \Big(1 - \frac{P_{l-1}(x)^2}{m^2}\Big)^{m^2\phi(i\delta)^2} \Big(1 - \frac{\delta^2}{m^2}\Big)^{-O(m^2\phi(i\delta)^2\eta)} \\
&\leq \exp\big(-P_{l-1}(x)^2\phi(i\delta)^2\big) \cdot (1+o(\eta)).
\end{aligned}$$

In addition, since $P_{l-1}(x)(1-3\eta) \leq x \leq \delta$, and by definition of $\gamma(i)$, we have

$$\begin{aligned}
\Big(1 - \frac{P_{l-1}(x)(1-3\eta)}{m}\Big)^{-2m\Phi(i\delta)\phi(i\delta)\cdot 1.02\Gamma(i)} &\leq \Big(1 - \frac{\delta}{m}\Big)^{-2m\Phi(i\delta)\phi(i\delta)\cdot 1.02\Gamma(i)} \leq 1 + 2.05\eta, \\
\Big(1 - \frac{P_{l-1}(x)^2(1-3\eta)^2}{m^2}\Big)^{-m^2\phi(i\delta)^2\cdot 1.02\Gamma(i)} &\leq \Big(1 - \frac{\delta^2}{m^2}\Big)^{-m^2\phi(i\delta)^2\cdot 1.02\Gamma(i)} \leq 1 + 1.03\eta.
\end{aligned}$$

Then by the fact that

$$p_l(x) = \exp\big(-P_{l-1}(x)^2\phi(i\delta)^2 - 2P_{l-1}(x)^2\Phi(i\delta)\phi(i\delta)\big),$$

we can conclude that

$$Q^* \leq p_l(x)(1+3\eta).$$

The argument for the lower bound on Q_* is similar. ■

7.4 When $2m\Phi(i\delta)\phi(i\delta)$ isn't an integer

We have defined the tree T_∞ so that for every node g at even distance from the root, the number of children of g that are sets of size 1 is exactly $\lceil 2m\Phi(i\delta)\phi(i\delta)\rceil$. We further made the simplifying assumption that $2m\Phi(i\delta)\phi(i\delta)$ is an integer. Reviewing our proof above, we needed this simplifying assumption in order to get a relatively simple solution to the differential equation in Section 7.1. Here we briefly explain how one can modify the proof above so as to handle the case where $2m\Phi(i\delta)\phi(i\delta)$ is not an integer.

The first step would be to take a random subtree of $T^*_{f,L}$. Let $\zeta \in [0.1, 0.9]$ be such that $\zeta \cdot 2m\Phi(i\delta)\phi(i\delta)$ is an integer. Keep every subtree of $T^*_{f,L}$ that is rooted at a set of size 1 with probability ζ. This gives us a random subtree of $T^*_{f,L}$. From now on we only care about this random subtree and so for brevity, we refer to this subtree by $T^*_{f,L}$. Using the fact that $\mathcal{E}^*$ holds, one can show the following. With probability $1 - n^{-\omega(1)}$, for every non-leaf node g at even distance from the root of $T^*_{f,L}$, the number of children of g that are sets of size 1 is $\zeta \cdot 2m\Phi(i\delta)\phi(i\delta)(1 \pm 1.02\Gamma(i))$ and the number of children of g that are sets of size 2 is as given in (7). Given that, we change the definition of T_∞ accordingly by asserting that for every node g at even distance from the root of T_∞, the number of children of g that are sets of size 1 is $\zeta \cdot 2m\Phi(i\delta)\phi(i\delta)$.

Having redefined the above trees, the second step is to redefine the distribution of the birthtimes of the edges in $T^*_{f,L}$ and T_∞. The birthtime of an edge that appears in a set of size 1 in $T^*_{f,L}$ or in T_∞ is redefined so that it is distributed uniformly at random in $[0, \zeta \cdot mn^{-1/2}]$, whereas the birthtime of an edge that appears in a set of size 2 in $T^*_{f,L}$ or in T_∞ remains uniformly distributed at random in $[0, mn^{-1/2}]$ as before.

The rest of the proof is straightforward. In particular, the statement of Lemma 7.2 is not changed. The only necessary other modifications are the obvious ones that follow from the above changes in the definition of $T^*_{f,L}$ and T_∞ and the definition of the birthtimes of the edges that appear in those trees.

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A Proof of Fact 4.2

Recall that $\frac{\sqrt{\pi}}{2}\mathrm{erfi}(\Phi(x)) = x$, where $\mathrm{erfi}(x)$ is the imaginary error function, given by, for example, $\mathrm{erfi}(x) = \frac{2}{\sqrt{\pi}}\sum_{j=0}^{\infty}\frac{x^{2j+1}}{j!(2j+1)}$. We have that $\mathrm{erfi}(x) \to \exp(x^2)/(\sqrt{\pi}x)$ as $x \to \infty$. Hence, it follows that as $x \to \infty$, $\Phi(x) \to \sqrt{\ln x}$ and $\phi(x) \to (2x\sqrt{\ln x})^{-1}$.

(i) We first upper bound $\phi(i\delta)$ and $\Phi(i\delta)$. We have that $\mathrm{erfi}(x) \geq 0$ if and only if $x \geq 0$. By the fact that $\frac{\sqrt{\pi}}{2}\mathrm{erfi}(\Phi(x)) = x$ we have $\mathrm{erfi}(\Phi(i\delta)) = 2i\delta/\sqrt{\pi} \geq 0$. Hence $\Phi(i\delta) \geq 0$. Therefore $\phi(i\delta) = \exp(-\Phi(i\delta)^2) \leq 1$. Next, note that $\mathrm{erfi}(x)$ is monotonically increasing with x. We also have by $\frac{\sqrt{\pi}}{2}\mathrm{erfi}(\Phi(x)) = x$ that $\mathrm{erfi}(\Phi(i\delta))$ is monotonically increasing with i. Hence $\Phi(i\delta)$ is monotonically increasing with i and so $\Phi(i\delta) \leq \Phi(I\delta)$. The upper bound on $\Phi(i\delta)$ now follows since $I\delta \sim n^{\varepsilon}$ and so $\Phi(I\delta) \sim \sqrt{\ln n^{\varepsilon}}$.

Next, we lower bound $\phi(i\delta)$ and $\Phi(i\delta)$ (for $i \geq 1$). Since $\Phi(i\delta)$ is monotonically increasing with i, we have that $\phi(i\delta)$ is monotonically decreasing with i. Therefore, it remains to show that $\phi(I\delta) = \Omega(\delta^{1.5})$ and $\Phi(\delta) = \Omega(\delta)$. The fact that $\phi(I\delta) = \Omega(\delta^{1.5})$ follows since $\phi(I\delta) \to 1/(2I\delta\sqrt{\ln I\delta})$. The fact that $\Phi(\delta) = \Omega(\delta)$ follows directly from the fact that $\frac{\sqrt{\pi}}{2}\mathrm{erfi}(\Phi(x)) = x$ and the definition of $\mathrm{erfi}(x)$.

(ii) By (i) we have $\delta\Phi(i\delta)\phi(i\delta) \leq \delta \ln n = o(1)$ and $\delta^2\phi(i\delta)^2 \leq \delta^2 = o(1)$. Hence $\gamma(i) = o(1)$. It also follows directly from the definition of $\gamma(i)$ and from the previous item that $\gamma(i) = \Omega(\delta^5)$.

We now bound $\Gamma(i)$. Since $\Gamma(i)$ is monotonically non-decreasing and $\Gamma(0) = n^{-30\varepsilon}$, it is enough to show that $\Gamma(I) \leq n^{-10\varepsilon}$. We do that by first showing that $\Gamma(\delta^{-1}\lfloor \ln\ln n\rfloor) \leq n^{-30\varepsilon+o(1)}$. For brevity, we shall assume below that $\lfloor \ln\ln n\rfloor = \ln\ln n$.

For every $0 \leq i \leq \delta^{-1}\ln\ln n$, $\Phi(i\delta) \leq \ln\ln n$ (crudely) and $\phi(i\delta) \leq 1$. Therefore, we have that for every $0 \leq i \leq \delta^{-1}\ln\ln n$,

$$\begin{aligned} \delta\Phi(i\delta)\phi(i\delta) &\leq \delta\ln\ln n, \text{ and} \\ \delta^2\phi(i\delta)^2 &\leq \delta\ln\ln n. \end{aligned}$$

Hence, for $0 \leq i \leq \delta^{-1}\ln\ln n$, $\gamma(i) \leq \delta\ln\ln n$ and so

$$\Gamma(\delta^{-1}\ln\ln n) \leq n^{-30\varepsilon}(1 + 10\delta\ln\ln n)^{\delta^{-1}\ln\ln n} = n^{-30\varepsilon+o(1)}.$$

Now, note that for every $\delta^{-1} \ln \ln n \le i \le I$,

$$\begin{aligned} \delta \Phi(i\delta)\phi(i\delta) &\le 0.6/i, \text{ and} \\ \delta^2 \phi(i\delta)^2 &\le 0.6/i, \end{aligned}$$

and this follows from the fact that for $\delta^{-1} \ln \ln n \le i \le I$, $\Phi(i\delta)\phi(i\delta) \sim 1/(2i\delta)$ and $\phi(i\delta) \le 1/(2i\delta)$. Hence, for $\delta^{-1} \ln \ln n \le i \le I$, $\gamma(i) \le 0.6/i$ and so we conclude that

$$\Gamma(I) \le n^{-30\varepsilon + o(1)} \prod_{1 \le i \le I} (1 + 6/i) \le n^{-30\varepsilon + o(1)} \cdot \exp(7 \ln I) \le n^{-10\varepsilon}.$$